

# UNIPOTENT REPRESENTATIONS AND THE DUAL PAIR CORRESPONDENCE

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*To Roger Howe with admiration*

## 1. INTRODUCTION

This paper describes some properties of the unipotent representations in relation to the  $\Theta$ –correspondence and rational functions on coadjoint nilpotent orbits in the Lie algebra.

Let  $\mathfrak{g}_{\mathbb{C}}$  be the complexification of a real reductive Lie algebra  $\mathfrak{g}$ , and  $G$  a real reductive group with Lie algebra  $\mathfrak{g}$  and maximal compact subgroup  $K \subset G$ .

**Definition 1.0.1.** *An irreducible  $(\mathfrak{g}, K)$ –module  $(\Pi, V)$  for a real reductive group  $G$  is called **unipotent** if*

- (1):  $\text{Ann } \Pi \subset U(\mathfrak{g})$  is a maximal primitive ideal,
- (2):  $(\Pi, V)$  is unitary.

Let  $(G_1, G_2)$  be pair of groups which form a dual reductive pair, and  $\Pi_1$  a unipotent representation of  $G_1$ . The question is when  $\Pi_1$  occurs in the  $\Theta$ –correspondence, as introduced and studied in the work of Roger Howe. The paper treats the case of complex groups viewed as real groups;  $\mathfrak{g}$  is the Lie algebra of a complex group viewed as a real group. A lot of the material is available for real groups, still in progress. The main reason for this restriction is that unipotent representations are classified in the case of complex groups in the sense that their Langlands parameters are explicitly given in [B1], and the Theta correspondence is also explicitly described in [AB1]. We mainly treat the cases of  $Sp(2n, \mathbb{C}) \times O(m, \mathbb{C})$ ;  $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$  is straightforward. The nature of the answer is that for any unipotent representation  $\Pi$ , there is a sequence

$$G_0 = G, G_1, \dots, G_r,$$

such that each  $(G_i, G_{i+1})$  is a dual pair, and unipotent representations  $\Pi_i$  so that  $(\Pi_i, \Pi_{i+1})$  occur in the  $\Theta$ –correspondence, and the last one  $\Pi_r$  is 1-dimensional. The precise conditions on  $\Pi_i$  and the  $G_i$  are given in Section 3, Theorem 3.5.1.

The second theme is the relation to regular functions on coadjoint orbits. To each unipotent representation one can associate a nilpotent orbit

$\mathcal{O} \subset \mathfrak{g}$  and a number  $m(\Pi, \mathcal{O})$  called the *Asymptotic Support* and *Multiplicity* respectively. We use the (equivalent) versions of *Associated Cycle* and *Multiplicity* in [V]. Let  $Unip(\mathcal{O})$  be the set of unipotent representations with asymptotic support  $\mathcal{O}$ . Let  $e \in \mathcal{O}$  be a representative,  $C_G(\mathcal{O}) := C_G(e)$  be the centralizer, and  $A(\mathcal{O}) := C_G(e)/C_G(e)^0$  be the component group. One of the main results in [V] is that there is an (algebraic) representation  $\psi(\Pi, \mathcal{O})$  of  $C_G(\mathcal{O})$  such that the multiplicity of  $\Pi$  is  $\dim \psi$ , and

$$\Pi|_{K_{\mathbb{C}}} = R(\mathcal{O}, \psi) - Y_{\psi}$$

where  $K$  is the maximal compact subgroup  $K \subset G$ , so  $K_{\mathbb{C}}$  is equivalent to  $G$ ,  $R(\mathcal{O}, \psi)$  the space of regular sections on  $\mathcal{O}$  transforming according to  $\psi$  viewed as  $G$ -module, and  $Y_{\psi}$  is a  $K_{\mathbb{C}}$ -representation with support in nilpotent orbits in the closure of  $\mathcal{O}$ , strictly smaller than  $\mathcal{O}$ . As already mentioned in [V], it is conjectured that there is a 1 – 1 correspondence  $\psi \longleftrightarrow \Pi_{\psi}$  between  $\widehat{A(\mathcal{O})}$  and  $Unip(\mathcal{O})$  such that

$$\Pi_{\psi}|_K \cong R(\mathcal{O}, \psi),$$

in particular  $Y_{\psi} = 0$ . We establish this conjecture for a large class of nilpotent orbits in the classical Lie groups. The relation follows for more general orbits from certain geometric properties of the resolution of nilpotent orbits for classical groups in [KP1]. We will pursue this in a later paper.

The correspondence between orbits and unipotent representations is conjectured to hold for general groups. The last sections investigate its validity for the simply connected groups  $Spin(n, \mathbb{C})$ , and the case of  $F_4$ . The groups of type  $E$  will be considered in a different paper.

Different properties of unipotent representations are considered in [Moe] and [MR]. There is very little overlap with the results in this paper.

One of the aims of the paper is to highlight the impact that Roger Howe's work had on my own work. I first met Roger Howe at a conference in Luminy in 1978. At the time I knew the work of Rallis and Schiffman and Kashiwara-Vergne on the dual pairs correspondence when one of the groups was compact. The case when neither group was compact seemed completely unreachable. I was stunned by the results that Roger presented for this latter case. Some ten years later, I understood enough to write a paper joint with J. Adams, [AB1], where we described the correspondence for complex groups in detail. Extensions of these results to some real classical groups appear in [AB2]. One of my students, Shu-Yen Pan, investigated the correspondence in the case of p-adic groups, and another student, Daniel Wong, investigated an extension of the Theta correspondence.

Along different lines, at the same time that I started my collaboration with Adams, I met and started to collaborate with Allen Moy. Another ten years later we gave a new proof of the Howe conjecture for p-adic groups.

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## 2. UNIPOTENT REPRESENTATIONS

In this section we review the basics of the representation theory of admissible representations of complex groups viewed as real groups.

**2.1. Complex Groups Viewed as Real Groups.** This material is taken from [V1]. Modules are all admissible  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules.

**Lemma 2.1.1.** *Let  $\mathfrak{g}$  be a complex Lie algebra, and let  $\mathfrak{g}_0$  be the same algebra viewed as a real Lie algebra. Then the complexification  $\mathfrak{g}_{\mathbb{C}}$  canonically identifies with*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_L + \mathfrak{g}_R.$$

*The summand  $\mathfrak{g}_L$  is isomorphic to  $\mathfrak{g}$ , and  $\mathfrak{g}_R$  to the complex conjugate algebra. The  $\star$ -antiautomorphism on  $\mathfrak{g}_{\mathbb{C}}$  interchanges the two summands.*

*Proof.* Let  $j$  be the multiplication by  $\sqrt{-1}$  on  $\mathfrak{g}$ . This is a real linear transformation on  $\mathfrak{g}_0$  and so defines a complex linear transformation  $J$  on  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 + i\mathfrak{g}_0$ , satisfying  $J(X + iY) = JX + iJY$  for  $X, Y \in \mathfrak{g}_0$ . Then

$$\begin{aligned} \mathfrak{g}_L &= \left\{ \frac{1}{2}(X - iJX) \mid X \in \mathfrak{g}_{\mathbb{C}} \right\}, \\ \mathfrak{g}_R &= \left\{ \frac{1}{2}(X + iJX) \mid X \in \mathfrak{g}_{\mathbb{C}} \right\}, \end{aligned}$$

are complex subalgebras of  $\mathfrak{g}_{\mathbb{C}}$ . The algebra  $\mathfrak{g}_L$  is isomorphic to  $\mathfrak{g}$  via

$$(\alpha + i\beta) - iJ(\alpha + i\beta) \mapsto \alpha + j\beta, \quad \alpha, \beta \in \mathfrak{g}_0.$$

The algebra  $\mathfrak{g}_R$  is isomorphic to  $\mathfrak{g}$  with conjugate linear multiplication.  $\square$

**2.2. Langlands Parameters.** We use the standard realizations of the classical groups, roots, positive roots and simple roots. Let

- $\theta$  Cartan involution,  $K$  the fixed points of  $\theta$ ,  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  the Cartan decomposition,
- $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$  a Borel subalgebra,
- $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$  a Cartan subalgebra,  $\mathfrak{t} \subset \mathfrak{k}$ ,  $\theta|_{\mathfrak{a}} = -Id$ ,
- $W$  the Weyl group of  $(\mathfrak{g}, \mathfrak{h})$ ,
- $X(\mu, \nu) = \text{Ind}_B^G(\mathbb{C}_{\mu} \otimes \mathbb{C}_{\nu})$  standard module,
- $L(\mu, \nu)$ , the unique subquotient containing  $V_{\mu} \in \widehat{K}$ ,
- $\lambda_L = (\mu + \nu)/2$  and  $\lambda_R = (-\mu + \nu)/2$ .

The parameters of unipotent representations have real  $\nu$ , so we will assume this in the rest of the paper.

**Theorem 2.2.1.**

- (1)  $L(\lambda_L, \lambda_R) \cong L(\lambda'_L, \lambda'_R)$  if and only if there is a  $w \in W$  such that  $w \cdot (\lambda_L, \lambda_R) = (\lambda'_L, \lambda'_R)$ .
- (2)  $L(\lambda_L, \lambda_R)$  is hermitian if and only if there is  $w \in W$  such that  $w \cdot (\mu, \nu) = (\mu, -\nu)$ .

We will write the parameter in column form as  $\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix}$  mainly for display reasons.

**2.3. Parameters of Unipotent Representations.** We rely on [BV2] and [B1]. For each  $\mathcal{O} \subset \mathfrak{g}$  we will give an infinitesimal character  $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ , and a set of  $(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})$  such that  $\{L(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})\}$  are the unipotent representations with asymptotic support  $\mathcal{O}$ . In all cases  $\lambda_{\mathcal{O}}$  and  $-\lambda_{\mathcal{O}}$  are in the same  $W$ -orbit.

**Main Properties of  $\lambda_{\mathcal{O}}$ .** Suppose  $\Pi$  is an irreducible representation with infinitesimal character  $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ . Then  $\lambda_{\mathcal{O}}$  and  $\Pi$  must satisfy:

- (1)  $\text{Ann } \Pi \subset U(\mathfrak{g})$  is the maximal primitive ideal  $\mathcal{I}_{\lambda_{\mathcal{O}}}$  with infinitesimal character  $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ ,
- (2)  $|\{\Pi : \text{Ann } \Pi = \mathcal{I}_{\lambda_{\mathcal{O}}}\}| = |\widehat{A(\mathcal{O})}|$ , where  $A(\mathcal{O})$  is the component group of the centralizer of an  $e \in \mathcal{O}$ ,
- (3)  $\Pi$  unitary.

**Remark 2.3.1.** *The component group  $A(\mathcal{O})$  depends on the isogeny class of  $G$ , which will be a classical group  $Sp(2n, \mathbb{C})$ ,  $SO(m, \mathbb{C})$  or  $O(m, \mathbb{C})$ .*

The notation is as in [B1]. The choices of  $\lambda_{\mathcal{O}}$  satisfying (3) rely on the determination of the unitary dual for classical groups in [B1]. For special orbits  $\mathcal{O}$  whose dual  $\mathcal{O}^{\vee}$  is even,  $\lambda_{\mathcal{O}}$  is half the semisimple element of the Lie triple corresponding to the dual orbit,  $\lambda_{\mathcal{O}} = h(\mathcal{O}^{\vee})/2$ . For the other orbits we need a case-by-case analysis. The parameter will always have integer and half-integer coordinates, and the corresponding system of integral coroots is maximal.

**Definition 2.3.2.** *A special orbit  $\mathcal{O}$  (in the sense of Lusztig) is called **stably trivial** if Lusztig's quotient  $\overline{A}(\mathcal{O})$  equals the full component group  $A(\mathcal{O})$ .*

For a definition and discussion of  $\overline{A}(\mathcal{O})$ , see [L], chapter 13.

The partitions in the next examples denote rows.

**Example 2.3.3.**  $\mathcal{O} = (2222) \subset sp(8)$  is stably trivial,  $A(\mathcal{O}) = \overline{A(\mathcal{O})} \cong \mathbb{Z}_2$ ,  $\lambda_{\mathcal{O}} = (2, 1, 1, 0)$ . In this case  $\mathcal{O}^{\vee}$  corresponds to the partition  $(531)$ , and  $\lambda_{\mathcal{O}} = h(\mathcal{O}^{\vee})/2$ .

$\mathcal{O} = (222) \subset sp(6)$  has dual orbit  $\mathcal{O}^{\vee}$  corresponding to  $(331)$  but is not stably trivial;  $A(\mathcal{O}) \cong \mathbb{Z}_2$ , while  $\overline{A(\mathcal{O})} \cong 1$ . In this case  $h(\mathcal{O}^{\vee})/2 = (1, 1, 0)$ , and for this infinitesimal character, conditions (1) and (3) are satisfied, but (2) is not satisfied. The choice of infinitesimal character in this case will be  $\lambda_{\mathcal{O}} = (3/2, 1/2, 1/2)$ . There are two parameters,

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ 3/2 & 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ 3/2 & 1/2 & -1/2 \end{pmatrix}$$

**2.4. Type A.** The group  $G$  is  $GL(n)$ . Nilpotent orbits are determined by their Jordan canonical form. An orbit is given by a partition, *i.e.* a sequence of numbers in decreasing order  $\mathcal{O} \longleftrightarrow (n_1, \dots, n_k)$  that add up to  $n$ . Let  $(m_1, \dots, m_l)$  be the dual partition. The component group of  $\mathcal{O}$  is trivial. The infinitesimal character is

$$\lambda_{\mathcal{O}} = \left( \frac{m_1 - 1}{2}, \dots, -\frac{m_1 - 1}{2}, \dots, \frac{m_l - 1}{2}, \dots, -\frac{m_l - 1}{2} \right).$$

The orbit is induced from the trivial orbit on the Levi component  $\mathfrak{m}$  of a parabolic subalgebra  $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$  with  $\mathfrak{m} = \mathfrak{gl}(m_1) \times \dots \times \mathfrak{gl}(m_l)$ . The corresponding unipotent representation is spherical and induced irreducible from the trivial representation on the same Levi component. All orbits are *special* and *stably trivial*.

**2.5. Type B.** We describe the case  $SO(2m + 1)$ . For  $O(2m + 1)$  there are twice the parameters, the parameters for  $SO$  are tensored with *sign*.

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). Then  $\mathcal{O}$  is parametrized by a partition  $\mathcal{O} \longleftrightarrow (n_1, \dots, n_k)$  of  $2m + 1$  such that every even entry occurs an even number of times. Let  $(m'_0, \dots, m'_{2p'})$  be the transpose partition (add an  $m'_{2p'} = 0$  if necessary, in order to have an odd number of terms). If  $\mathcal{O}$  is represented by a tableau, these are the sizes of the columns in decreasing order. If there are any  $m'_{2j} = m'_{2j+1}$ , then pair them together and remove them from the partition. Then relabel and pair up the remaining columns  $(m_0)(m_1, m_2) \dots (m_{2p-1} m_{2p})$ . The members of each pair have the same parity and  $m_0$  is odd.  $\lambda_{\mathcal{O}}$  is given by the coordinates

$$(1) \quad \begin{aligned} (m_0) &\longleftrightarrow \left( \frac{m_0 - 2}{2}, \dots, \frac{1}{2} \right), \\ (m'_{2j} = m'_{2j+1}) &\longleftrightarrow \left( \frac{m'_{2j} - 1}{2}, \dots, -\frac{m'_{2j} - 1}{2} \right) \\ (m_{2i-1} m_{2i}) &\longleftrightarrow \left( \frac{m_{2i-1}}{2}, \dots, -\frac{m_{2i} - 2}{2} \right). \end{aligned}$$

In case  $m'_{2j} = m'_{2j+1}$ ,  $\mathcal{O}$  is induced from an orbit

$$\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} = \mathfrak{so}(\ast) \times \mathfrak{gl}\left(\frac{m'_{2j} + m'_{2j+1}}{2}\right)$$

where  $\mathfrak{m}$  is the Levi component of a parabolic subalgebra  $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ .  $\mathcal{O}_{\mathfrak{m}}$  is the trivial nilpotent on the  $\mathfrak{gl}$ -factor. The component groups satisfy  $A_G(\mathcal{O}) \cong A_M(\mathcal{O}_{\mathfrak{m}})$ . Each unipotent representation is unitarily induced from a unipotent representation attached to  $\mathcal{O}_{\mathfrak{m}}$ .

Similarly if some  $m_{2i-1} = m_{2i}$ , then  $\mathcal{O}$  is induced from a

$$\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} \cong \mathfrak{so}(\ast) \times \mathfrak{gl}\left(\frac{m_{2i-1} + m_{2i}}{2}\right) \quad (0) \quad \text{on the gl-factor.}$$

$A_G(\mathcal{O}) \not\cong A_M(\mathcal{O}_{\mathfrak{m}})$ , but each unipotent representation is (not necessarily unitarily) induced irreducible from a representation on the Levi component  $\mathfrak{m}$ , unipotent on  $so(*)$ , and a character on the  $\mathfrak{gl}$ -factor.

The *stably trivial* orbits are the ones such that every odd sized part appears an even number of times, except for the largest size. An orbit is called triangular if it has partition

$$\mathcal{O} \longleftrightarrow (2m+1, 2m-1, 2m-1, \dots, 3, 3, 1, 1).$$

We give the explicit Langlands parameters of the unipotent representations. There are  $|A_G(\mathcal{O})|$  distinct representations. Let

$$(\underbrace{k, \dots, k}_{r_k}, \dots, \underbrace{1, \dots, 1}_{r_1})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers  $r_{2i}$  are even. The reductive part of the centralizer (when  $G$  is the orthogonal group) of the nilpotent element is a product of  $O(r_{2i+1})$ , and  $Sp(r_{2j})$ .

The columns are paired as in (1). The pairs  $(m'_{2j} = m'_{2j+1})$  contribute to the spherical part of the parameter,

$$(2) \quad (m'_{2j} = m'_{2j+1}) \longleftrightarrow \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \\ \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \end{pmatrix}.$$

The singleton  $(m_0)$  contributes to the spherical part,

$$(3) \quad (m_0) \longleftrightarrow \begin{pmatrix} \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \\ \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \end{pmatrix}.$$

Let  $(\eta_1, \dots, \eta_p)$  with  $\eta_i = \pm 1$ , one for each  $(m_{2i-1}, m_{2i})$ . An  $\eta_i = 1$  contributes to the spherical part of the parameter, with coordinates as in (2) and (3). An  $\eta_i = -1$  contributes

$$(4) \quad \begin{pmatrix} \frac{m_{2i-1}-1}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}}{2} & , & \cdots & , & -\frac{m_{2i}-2}{2} \\ \frac{m_{2i-1}-1}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}-2}{2} & , & \cdots & , & -\frac{m_{2i}}{2} \end{pmatrix}.$$

If  $m_{2p} = 0$ ,  $\eta_p = 1$  only for  $SO$ .

## 2.6. Explanation.

- (1) Odd sized rows contribute a  $\mathbb{Z}_2$  to  $A(\mathcal{O})$ , even sized rows a 1.
- (2) When there are no  $m'_{2j} = m'_{2j+1}$ , every row size occurs. The inequalities

$$\dots (m_{2i-1} \geq m_{2i}) > (m_{2i+1} \geq m_{2i+2}) \dots$$

imply that there are  $m_{2i} - m_{2i+1}$  rows of size  $2i + 1$ . Each pair  $(m_{2i-1} \geq m_{2i})$  contributes exactly 2 parameters corresponding to the  $\mathbb{Z}_2$  in  $A(\mathcal{O})$ .

- (3) The pairs  $(m'_{2j} = m'_{2j+1})$  lengthen the sizes of the rows without changing their parity. The component group does not change, they do not affect the number of parameters.

As already mentioned, when  $G = O(2m + 1, \mathbb{C})$  the unipotent representations are obtained from those of  $SO(2m, \mathbb{C})$  by lifting them to  $O(2m, \mathbb{C})$ , and also tensoring with  $sgn$ .

**2.7. Type C.** A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition  $\mathcal{O} \longleftrightarrow (n_1, \dots, n_k)$  of  $2n$  such that every odd part occurs an even number of times. Let  $(c'_0, \dots, c'_{2p'})$  be the dual partition (add a  $c'_{2p'} = 0$  if necessary in order to have an odd number of terms). As in type B, these are the sizes of the columns of the tabelau corresponding to  $\mathcal{O}$ . If there are any  $c'_{2j-1} = c'_{2j}$  pair them up and remove them from the partition. Then relabel and pair up the remaining columns  $(c_0 c_1) \dots (c_{2p-2} c_{2p-1}) (c_{2p})$ . The members of each pair have the same parity. The last one,  $c_{2p}$ , is always even. Then form a parameter

$$(5) \quad (c'_{2j-1} = c'_{2j}) \longleftrightarrow \left( \frac{c_{2j} - 1}{2}, \dots, -\frac{c_{2j} - 1}{2} \right),$$

$$(6) \quad (c_{2i} c_{2i+1}) \longleftrightarrow \left( \frac{c_{2i}}{2}, \dots, -\frac{c_{2i+1} - 2}{2} \right),$$

$$(7) \quad c_{2p} \longleftrightarrow \left( \frac{c_{2p}}{2}, \dots, 1 \right).$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

An orbit is called triangular if it corresponds to the partition  $(2m, 2m, \dots, 4, 4, 2, 2)$ .

We give a parametrization of the unipotent representations in terms of their Langlands parameters. There are  $|A_G(\mathcal{O})|$  representations.

Let

$$\underbrace{(k, \dots, k)}_{r_k}, \dots, \underbrace{(1, \dots, 1)}_{r_1}$$

be the rows of the Jordan form of the nilpotent orbit. The numbers  $r_{2i+1}$  are even. The reductive part of the centralizer of the nilpotent element is a product of  $Sp(r_{2i+1})$ , and  $O(r_{2j})$ .

The elements  $(c'_{2j-1} = c'_{2j})$  and  $c_{2p}$  contribute to the spherical part of the parameter as in (2) and (3). Let  $(\eta_1, \dots, \eta_p)$  be such that  $\eta_i = \pm 1$ , one for each  $(c_{2i}, c_{2i+1})$ . An  $\eta_i = 1$  contributes to the spherical part, according to the infinitesimal character. An  $\eta_i = -1$  contributes

$$(8) \quad \left( \begin{array}{ccccccc} \frac{c_{2i}}{2} & , & \cdots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}}{2} & \cdots & , & -\frac{c_{2i+1}-2}{2} \\ \frac{c_{2i}}{2} & , & \cdots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}-2}{2} & \cdots & , & -\frac{c_{2i+1}}{2} \end{array} \right).$$

The explanation is similar to type B.

**2.8. Type D.** We treat the case  $G = SO(2m)$ . A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition  $\mathcal{O} \longleftrightarrow (n_1, \dots, n_k)$  of  $2m$  such that every even part occurs an even number of times. Let  $(m'_0, \dots, m'_{2p'-1})$  be the dual partition (add a  $m'_{2p'-1} = 0$  if necessary), the sizes of the columns of the tableau corresponding to  $\mathcal{O}$ . If there are any  $m'_{2j} = m'_{2j+1}$  pair them up and remove from the partition. Then pair up the remaining columns  $(m_0, m_{2p-1})(m_1, m_2) \dots (m_{2p-3}, m_{2p-2})$ . The members of each pair have the same parity and  $m_0, m_{2p-1}$  are both even. The infinitesimal character is

$$(9) \quad \begin{aligned} (m'_{2j} = m'_{2j+1}) &\longleftrightarrow \left( \frac{m'_{2j} - 1}{2}, \dots, -\frac{m'_{2j} - 1}{2} \right) \\ (m_0 m_{2p-1}) &\longleftrightarrow \left( \frac{m_0 - 2}{2}, \dots, -\frac{m_{2p-1}}{2} \right), \\ (m_{2i-1} m_{2i}) &\longleftrightarrow \left( \frac{m_{2i-1}}{2}, \dots, -\frac{m_{2i} - 2}{2} \right) \end{aligned}$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. An exception occurs for  $G = SO(2m)$  when the partition is formed of pairs  $(m'_{2j} = m'_{2j+1})$  only. In this case there are two nilpotent orbits corresponding to the partition. There are also two nonconjugate Levi components of the form  $gl(m'_0) \times gl(m'_2) \times \dots \times gl(m'_{2p'-2})$  of parabolic subalgebras. There are two unipotent representations each induced irreducible from the trivial representation on the corresponding Levi component.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

A nilpotent orbit is triangular if it corresponds to the partition  $(2m-1, 2m-1, \dots, 3, 3, 1, 1)$ .

The parametrization of the unipotent representations follows types B,C, with the pairs  $(m'_{2j} = m'_{2j+1})$  and  $(m_0, m_{2p-1})$  contributing to the spherical part of the parameter only. Similarly for  $(m_{2i-1}, m_{2i})$  with  $\epsilon_i = 1$  spherical only, while  $\epsilon_i = -1$  contributes analogous to (4) and (8).

The explanation parallels that for types B,C.

When  $G = O(2m, \mathbb{C})$  the unipotent representations are obtained from those of  $SO(2m, \mathbb{C})$  by lifting them to  $O(2m, \mathbb{C})$ , and also tensoring with *sgn*. In the case when all  $m'_{2j} = m'_{2j+1}$  the representations associated to the two nilpotent orbits have the same lift, and it is invariant under tensoring with *sgn*. Otherwise tensoring with *sgn* gives inequivalent unipotent representations.

### 3. THETA CORRESPONDENCE

We deal with the complex pairs  $G_1 \times G_2$  where one group is orthogonal the other symplectic. The results are from [AB1]. Let  $V_i$  for  $i = 1, 2$  be



spaces endowed with nondegenerate forms, one symplectic the other orthogonal. Then  $\mathcal{W} = V_1 \otimes V_2$ , is symplectic, and  $G_1 \times G_2 := G(V_1) \times G(V_2)$  is a dual pair. Up to isomorphism,  $(G_1, G_2)$  is  $(O(n, \mathbb{C}), Sp(2m, \mathbb{C}))$  or  $(Sp(2m, \mathbb{C}), O(n, \mathbb{C}))$ . Let  $\tau = 0, 1$  depending whether  $n$  (for the orthogonal group) is even or odd.

**3.1. Complex Pairs.** Let  $(V_0, \langle \cdot, \cdot \rangle_0)$  be a real symplectic vector space. We can view  $\langle \cdot, \cdot \rangle_0$  as a linear map  $\mathcal{J}_0 : V_0 \rightarrow V'_0$  ( $V'_0$  the linear dual of  $V_0$ ) satisfying  $\mathcal{J}_0^t = -\mathcal{J}_0$ , so that the symplectic form is given by

$$(10) \quad \langle v_1, v_2 \rangle_0 = (\mathcal{J}_0 v_2)(v_1).$$

Let  $V_{\mathbb{C}} = V_0 + iV_0$  be the complexification of  $V_0$ , and  $\langle \cdot, \cdot \rangle$  be the complexification of  $\langle \cdot, \cdot \rangle_0$ . It satisfies

$$(11) \quad \langle v_1 + iv_2, w_1 + iw_2 \rangle = (\langle v_1, w_1 \rangle_0 - \langle v_2, w_2 \rangle_0) + i(\langle v_1, w_2 \rangle_0 + \langle v_2, w_1 \rangle_0)$$

The complex symplectic Lie algebra  $\mathfrak{g}_0 := sp(V_{\mathbb{C}})$  is the algebra preserving  $\langle \cdot, \cdot \rangle$ . Let  $\mathcal{V} = V_0 \oplus V_0$  be the real vector space identified with  $V_{\mathbb{C}}$  in the usual way,  $v_1 + iv_2 \longleftrightarrow (v_1, v_2)$ . An element  $a = \alpha + i\beta \in sp(V_{\mathbb{C}})$  is then

$$(12) \quad \alpha + i\beta \longleftrightarrow \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

The real part and imaginary part of  $\langle \cdot, \cdot \rangle$  are symplectic (nondegenerate) forms on  $\mathcal{V}$ ; denote them by  $\langle \cdot, \cdot \rangle_{re}$  and  $\langle \cdot, \cdot \rangle_{im}$ . In terms of skew maps from  $\mathcal{V}$  to  $\mathcal{V}'$ , they are

$$(13) \quad \begin{aligned} \langle \cdot, \cdot \rangle_{re} &\longleftrightarrow \begin{bmatrix} \mathcal{J}_0 & 0 \\ 0 & -\mathcal{J}_0 \end{bmatrix}, \\ \langle \cdot, \cdot \rangle_{im} &\longleftrightarrow \begin{bmatrix} 0 & \mathcal{J}_0 \\ \mathcal{J}_0 & 0 \end{bmatrix}. \end{aligned}$$

View  $sp(V_{\mathbb{C}})$  as a real Lie algebra. Then  $sp(V_{\mathbb{C}})$  embeds in  $(sp(\mathcal{V}), \langle \cdot, \cdot \rangle_{re, im})$  via formula (12). We choose  $\langle \cdot, \cdot \rangle_{re}$ , and note that  $sp(V_{\mathbb{C}})$ ,  $sp(\mathcal{V})$  are invariant under transpose, and the inclusion  $sp(V) \subset sp(\mathcal{V})$  commutes with the transpose map. We will view  $sp(V)$  as the Lie subalgebra of  $sp(\mathcal{V})$  under the inclusion (12).

The Cartan decomposition of (the real Lie algebra)  $\mathfrak{g}_0 := sp(V_{\mathbb{C}})$  is

$$(14) \quad \begin{aligned} \mathfrak{g}_0 &= \mathfrak{k}_0 + \mathfrak{s}_0, \\ \mathfrak{k}_0 &= \{\alpha + i\beta : (\alpha + i\beta) + (\alpha - i\beta)^t = 0\}, \\ \mathfrak{s}_0 &= \{\alpha + i\beta : (\alpha + i\beta) - (\alpha - i\beta)^t = 0\}. \end{aligned}$$

Similarly the Cartan decomposition of  $\mathfrak{g}_{\mathcal{V}} := sp(\mathcal{V})$  is

$$(15) \quad \begin{aligned} \mathfrak{g}_{\mathcal{V}} &= \mathfrak{k}_{\mathcal{V}} + \mathfrak{s}_{\mathcal{V}}, \\ \mathfrak{k}_{\mathcal{V}} &= \{A \in sp(\mathcal{V}) : A + A^t = 0\}, \\ \mathfrak{s}_{\mathcal{V}} &= \{A \in sp(\mathcal{V}) : A - A^t = 0\}. \end{aligned}$$

In particular,  $\mathfrak{k}_0 \subset \mathfrak{k}_{\mathcal{V}}$  and  $\mathfrak{s}_0 \subset \mathfrak{s}_{\mathcal{V}}$ .

3.1.1. *A Variant.* Let  $(V, \langle \cdot, \cdot \rangle)$  be a symplectic complex space with form corresponding to  $\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ . Then  $sp(2n, \mathbb{C})$  embeds in  $sp(4n, \mathbb{C})$  with the usual symplectic form as

$$\begin{bmatrix} \alpha & \beta \mathcal{J} \\ -\mathcal{J} \beta & -\alpha^t \end{bmatrix}$$

where  $\alpha, \beta \in sp(2n)_c$  is the compact real form of  $sp(2n, \mathbb{C})$ . Multiplication by  $\sqrt{-1}$  corresponds to

$$m_{\sqrt{-1}} : \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha^t \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \alpha \mathcal{J} \\ -\mathcal{J} \alpha & 0 \end{bmatrix}.$$

The complexification  $sp(2n, \mathbb{C})_c \subset sp(4n, \mathbb{C})$  is the same, replace  $\alpha, \beta \in sp(2n, \mathbb{R})$  by  $\alpha, \beta \in sp(2n, \mathbb{C})$ .

3.2. **Oscillator Representation.** Let  $\Omega_{\mathcal{V}} = \Omega_+ + \Omega_-$  be the oscillator representation of  $sp(\mathcal{V}) = sp(4n, \mathbb{R})$ .

The following is well known (and straightforward).

**Theorem 3.2.1.**

- (1) *The pairs  $(O(m, \mathbb{C}), Sp(2n, \mathbb{C})) \subset Sp(2mn, \mathbb{C}) \subset Sp(4mn, \mathbb{R})$  are dual pairs.*
- (2) *The restrictions to  $sp(2n, \mathbb{C})$  of  $\Omega_{\pm}$  are irreducible and equal to the two representations of  $sp(2n, \mathbb{C})$  corresponding to the minimal non-trivial nilpotent orbit.*

3.3. **Infinitesimal Character.**

**Proposition 3.3.1.** *Suppose  $\pi_1$  corresponds to  $\pi_2$  in the dual pair correspondence for  $(G_1, G_2)$ . Let  $(\lambda_1, \lambda'_1)$  be the infinitesimal character of  $\pi_1$ . Write the infinitesimal character of  $\pi_2$  as  $(\lambda_2, \lambda'_2)$ . Then for  $\lambda_2$  and  $\lambda'_2$  we may take  $\lambda_2 = \lambda_1 \cdot \tilde{\lambda}$  and  $\lambda'_2 = \lambda'_1 \cdot \tilde{\lambda}$  with  $\tilde{\lambda}$  as follows:*

- (1)  $(O(m), Sp(2n)), [\frac{m}{2}] \leq n$ :  $\tilde{\lambda} = (n - m/2, n - m/2 - 1, \dots, 1 - \tau/2)$ ,
- (2)  $(Sp(2m), O(n)), m \leq [\frac{n}{2}]$ :  $\tilde{\lambda} = (n/2 - m - 1, n/2 - m - 2, \dots, \tau/2)$ ,
- (3)  $(GL(m), GL(n)), m \leq n$ :  $\tilde{\lambda} = \frac{1}{2}(n - m - 1, n - m - 3, \dots, -n + m + 1)$ .

Here  $\cdot$  indicates concatenation of sequences.

3.4. **Langlands Parameters.**  $K$ -types for  $O(n)$  are parametrized as in Weyl's work using the standard embedding  $O(n) \subset U(n)$ . An irreducible representation of  $O(n)$  is parametrized by

$$(a_1, \dots, a_k, 0 \dots 0; \epsilon)$$

with  $\dots a_i \geq a_{i+1} \geq \dots a_k > 0$  integers and  $\epsilon = 0, 1$  so that the representation is the  $O(n)$ -irreducible component generated by the highest weight of

the representation of  $U(n)$  with highest weight

$$(a_1, \dots, a_k, \underbrace{1, \dots, 1}_{n-2k}, 0, \dots, 0).$$

The basic cases for the correspondence are summarized in the next proposition. The general case is the next theorem.

**Proposition 3.4.1.** *[Proposition 2.1, [AB1], Basic Cases (Type I)]*

- (1) *Let  $\text{triv}$  be the trivial representation of  $O(m, \mathbb{C})$ . Then for any  $n \geq 0$ ,  $\Theta(\text{triv})$  is the unique irreducible spherical representation of  $Sp(2n, \mathbb{C})$  with infinitesimal character given by Proposition 3.3.1. Thus*

*$\Theta(\text{triv}) = L(0, \nu)$  with  $\nu = (m-2, m-4, \dots, m-2n)$ . In terms of  $\lambda_L, \lambda_R$ , the parameter is*

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m/2 - 1, \dots, m/2 - n \\ m/2 - 1, \dots, m/2 - n \end{pmatrix}$$

- (2) *Let  $\text{triv}$  be the trivial representation of  $Sp(2m, \mathbb{C})$ .*

*(a) For any even  $n \geq 0$ ,  $\Theta(\text{triv})$  is the unique irreducible spherical representation of  $O(n, \mathbb{C})$  with infinitesimal character given by Proposition 3.3.1. Thus  $\Theta(\text{triv}) = L(0, \nu)$  with*

$$\nu = (2m, 2m-2, \dots, 2m-n+2).$$

*In terms of  $\lambda_L, \lambda_R$  the parameter is*

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m, m-1, \dots, m-n/2+1 \\ m, m-1, \dots, m-n/2+1 \end{pmatrix}$$

*(b) If  $n$  is odd, then  $\text{triv}$  occurs in the correspondence with  $O(n, \mathbb{C})$  if and only if  $n > 2m$ , and the same conclusion as in (a) holds with*

$$\nu = (2m, 2m-2, \dots, 2, -1, -3, \dots, 2m-n+2).$$

*In terms of  $\lambda_L, \lambda_R$ , the parameter is*

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m, m-1, \dots, 1, -1/2, \dots, m-n/2+1 \\ m, m-1, \dots, 1, -1/2, \dots, m-n/2+1 \end{pmatrix}$$

- (3) *The  $\text{sgn}$  representation of  $O(m, \mathbb{C})$  occurs in the representation correspondence with  $Sp(2n, \mathbb{C})$  if and only if  $n \geq m$ .*

*(a) For  $n = m$ ,  $\Theta(\text{sgn})$  is the unique irreducible representation of  $Sp(2m, \mathbb{C})$  with lowest  $K$ -type equal to the  $K$ -type pairing with the  $\text{sgn}$  representation of  $O(m)$  (cf. Proposition 1.4 in [AB1]), and infinitesimal character given by Proposition 3.3.1. Thus  $\Theta(\text{sgn}) = L(\mu, \nu)$  with*

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ m-1 & m-3 & \dots & -m+1 \end{pmatrix}$$

In terms of  $\lambda_L, \lambda_R$ , the parameter is

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m/2 + 1/2, \dots, -m/2 + 1/2 \\ m/2 - 1/2, \dots, -m/2 - 1/2 \end{pmatrix}$$

(b) For  $n > m$ , let  $P$  be a parabolic subgroup of  $Sp(2n, \mathbb{C})$  with Levi factor  $M = GL(n - m, \mathbb{C}) \times Sp(2m, \mathbb{C})$ . Then  $\Theta(\text{sgn})$  is the unique irreducible subquotient of

$$\text{Ind}_P^{Sp(2n, \mathbb{C})}(|\det|^{n+1} \otimes \Theta_m(\text{sgn}))$$

containing the lowest  $K$ -type of this induced representation. Here  $\Theta_m$  denotes the  $\Theta$ -lift from  $O(m, \mathbb{C})$  to  $Sp(2m, \mathbb{C})$ . Explicitly:  $\Theta(\text{sgn}) = L(\mu, \nu)$  with

$$\begin{aligned} \mu &= (\overbrace{1, \dots, 1}^m, 0, \dots, 0), \\ \nu &= (\overbrace{m-1, m-3, \dots, -m+1}^m, 2n-m, 2n-m-2, \dots, m+2). \end{aligned}$$

In terms of  $\lambda_L, \lambda_R$ , the parameter is

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m/2 - 1/2, \dots, -m/2 + 3/2, n-m, \dots, m/2 + 1 \\ m/2 - 3/2, \dots, -m/2 + 1/2, n-m, \dots, m/2 + 1 \end{pmatrix}$$

**Theorem 3.4.2** ([AB1] Theorem 2.8: Explicit dual pair correspondence (Type I)). Fix  $\tau = 0, 1$  and consider a family of dual pairs  $(G_1(m), G_2(n)) = (O(2m + \tau, \mathbb{C}), Sp(2n, \mathbb{C}))$ . Fix  $m$ , let  $G_1 = G_1(m)$ , and let  $\pi_1 = L(\mu_1, \nu_1)$  be an irreducible representation of  $G_1$ .

Define the integer  $k = k[\mu_1]$  by writing  $\mu_1 = (a_1, \dots, a_k, 0, \dots, 0; \epsilon)$  with  $a_1 \geq a_2 \geq \dots \geq a_k > 0$ . Write  $\nu_1 = (b_1, \dots, b_m)$ , and define the integer  $0 \leq q = q[\mu_1, \nu_1] \leq m - k$  to be the largest integer such that  $2q - 2 + \tau, 2q - 4 + \tau, \dots, \tau$  all occur (in any order) in  $\{\pm b_{k+1}, \pm b_{k+2}, \dots, \pm b_m\}$ . After possibly conjugating by the stabilizer of  $\mu_1$  in  $W$ , we may write

$$\begin{aligned} \mu_1 &= (\overbrace{a_1, \dots, a_k}^k, \overbrace{0, \dots, 0}^{m-q-k}, \overbrace{0, \dots, 0}^q; \epsilon) \\ \nu_1 &= (\overbrace{b_1, \dots, b_k}^k, \overbrace{b_{k+1}, \dots, b_{m-q}}^{m-q-k}, \overbrace{2q-2+\tau, 2q-4+\tau, \dots, \tau}^q). \end{aligned}$$

Let  $\mu'_1 = (a_1, \dots, a_k)$ ,  $\nu'_1 = (b_1, \dots, b_k)$ , and  $\nu''_1 = (b_{k+1}, \dots, b_{m-q})$ .

Then for  $n \geq n(\pi_1) = m - \epsilon q + \frac{1-\epsilon}{2}\tau$ ,  $\Theta_n(\pi_1) = L(\mu_2, \nu_2)$ , where

$$\begin{aligned} \mu_2 &= (\mu'_1, \overbrace{1, \dots, 1}^{\frac{1-\epsilon}{2}(2q+\tau)}, 0, \dots, 0) \\ \nu_2 &= (\nu'_1, \overbrace{2q-1+\tau, 2q-3+\tau, \dots, 2\epsilon q+1+\epsilon\tau}^{\frac{1-\epsilon}{2}(2q+\tau)}, \nu''_1, \\ &\quad 2n-2m-\tau, 2n-2m-2-\tau, \dots, -\epsilon(2q+\tau)+2). \end{aligned}$$

If  $\lambda_L, \lambda_R$  are the parameter of  $\pi_1$ , then the parameter of  $\Theta(\pi_1)$  is

$$\begin{pmatrix} \lambda_L, q - \tau/2 + 1/2, \dots, \epsilon q + \epsilon\tau/2 + 3/2, n - m - \tau/2, \dots, -\epsilon q - \epsilon\tau/2 + 1 \\ \lambda_R, q - \tau/2 - 1/2, \dots, \epsilon q + \epsilon\tau/2 + 1/2, n - m - \tau/2, \dots, -\epsilon q - \epsilon\tau/2 + 1 \end{pmatrix}$$

Note: The lowest K-type  $\mu_1 = (a_1, \dots, a_k, 0, \dots, 0; \epsilon)$  for  $\pi_1$  is degree-lowest in  $\pi_1$  if  $\epsilon = +1$ . If  $\epsilon = -1$  the degree-lowest K-type of  $\pi_1$  is

$$(a_1, \dots, a_k, \overbrace{1, \dots, 1}^r, 0, \dots, 0; \eta)$$

where

$$(r, \eta) = \begin{cases} (2q + \tau, 1) & \text{if } 2q + \tau \leq m - k, \\ (2(m - k) - 2q, -1) & \text{if } m - k < 2q + \tau \leq 2(m - k) + \tau. \end{cases}$$

**3.5. Main Result.** Restrict attention to the cases when the nilpotent orbit  $\mathcal{O}$  has columns

- (B):  $(m_0)(m_1, m_2) \dots (m_{2p-1}, m_{2p})$  with  $m_{2k} > m_{2k+1}$ ,
- (C):  $(c_0, c_1) \dots (c_{2p-2}, c_{2p-1})(c_{2p})$  with  $c_{2j-1} > c_{2j}$ ,
- (D):  $(m_0, m_{2p+1})(m_1, m_2) \dots (m_{2p-1}, m_{2p})$  with  $m_{2j} > m_{2j+1}$ .

To each such nilpotent orbit we associate a sequence of dual pairs as follows. Let  $(V_k, \epsilon_k)$  be a symplectic space if  $\epsilon_k = -1$ , orthogonal if  $\epsilon_k = 1$ ,  $k = 0, \dots, 2p$ .  $\epsilon_0$  is the same as the type of the Lie algebra,  $\dim V_0$  is the sum of the columns. Let  $(V_k, \epsilon_k)$  be the space with dimension the sum of the lengths of the columns labelled  $\geq k$ , and set  $\epsilon_{k+1} = -\epsilon_k$ . Then

$$(V_k, V_{k+1})$$

gives rise to a dual pair.

**Theorem 3.5.1.** *The unipotent representations attached to  $\mathcal{O}_k$  are all  $\Theta$ -lifts of the unipotent representations attached to  $\mathcal{O}_{k+1}$ . More precisely, it is enough to describe the passage from  $\mathcal{O}_1$  to  $\mathcal{O}_0$ .*

- The infinitesimal character for  $\mathcal{O}_0$  is obtained from  $\lambda_{\mathcal{O}_1}$  by the procedure in proposition 3.3.1; the resulting infinitesimal character is  $\lambda_{\mathcal{O}_0}$ .
- $\epsilon_0 = -1$ . There is a 1-1 correspondence between unipotent representations of  $Sp(V_1)$  attached to  $\mathcal{O}_1$  and unipotent representations of  $SO(V_0)$  attached to  $\mathcal{O} = \mathcal{O}_0$ .
- $\epsilon_0 = 1$ . There is a 1-1 correspondence between unipotent representations of  $O(V_1)$  attached to  $\mathcal{O}_1$  and unipotent representations of  $Sp(V_0)$  attached to  $\mathcal{O} = \mathcal{O}_0$ .

*Proof.* The relation between  $\mathcal{O}_1$  and  $\mathcal{O}_0$  is that one adds a column longer than the longest column of  $\mathcal{O}_1$ . This adds one to the existing rows of  $\mathcal{O}_1$  and adds some rows of size 1. When passing from  $sp(*)$  to  $so(*)$ , the component group acquires another  $\mathbb{Z}_2$ . When passing from  $so(*)$  to  $sp(*)$ , the component group does not change.

If  $\mathcal{O}_1$  is type C then  $\mathcal{O}_0$  is type B, and we add a column  $m_0$  which must be of odd length. The infinitesimal character is augmented by  $(m_0/2, \dots, 1/2)$  conforming to 3.3.1. There are two cases:

- (1)  $c_{2p} = 0$ . In this case  $c_0 \rightarrow m_1, \dots, c_{2p-1} \rightarrow m_{2p}$ . So the pairing of the columns of  $\mathcal{O}_0$  matches  $(m_0)(c_0, c_1) \dots (c_{2p-2}, c_{2p-1})$  and  $\Theta$  gives a 1-1 correspondence between parameters for  $\mathcal{O}_1$  and  $\mathcal{O}_0$ .
- (2)  $c_{2p} \neq 0$ . In this case,  $c_{2p}$  is even. Again  $c_0 \rightarrow m_1, \dots, c_{2p-1} \rightarrow m_{2p}$ , but  $c_{2p} \rightarrow m_{2p+1}$  and we have to add  $m_{2p+2} = 0$ . The pairing of columns for  $\mathcal{O}_0$  is  $(m_0)(c_1, c_2) \dots (c_{2p-2}, c_{2p-1})(c_{2p}, 0)$ . Since  $c_{2p} > 0$  is even, the last pair does not contribute any unipotent representations.

In both cases

$$(\eta_1, \dots, \eta_p) \longleftrightarrow (\eta_1, \dots, \eta_p).$$

If the pair is from type  $C$  to type  $D$ , a column  $m_0$  is added, and the infinitesimal character matches Proposition 3.3.1.  $c_0, \dots, c_{2p}$  are changed to  $m_1, \dots, m_{2p+1}$ . The pairing of the columns of  $\mathcal{O}_0$  is

$$(m_0, c_{2p})(c_1, c_2) \dots (c_{2p-2}, c_{2p-1}).$$

A parameter corresponding to a  $(\eta_1, \dots, \eta_p)$  goes to the corresponding one with  $(\eta_1, \dots, \eta_p)$  for type  $D$ .

The correspondence for parameters of type  $B, D$  with type  $C$  when the lowest  $K$ -type is with a  $+$  is analogous to type  $C$  to type  $B, D$  above. The cases when the lowest  $K$ -type is with a  $-$  are as follows. In all cases the infinitesimal characters conform to proposition 3.3.1.

For type  $B$  to type  $C$ , an odd column  $c_0$  larger than  $m_0$  is added, and  $m_0 \rightarrow c_1, \dots, m_{2p} \rightarrow c_{2p+1}$ , and we must add a  $c_{2p+2} = 0$ . The pairing of the columns is

$$(c_0, m_0)(m_1, m_2) \dots (m_{2p-1}, m_{2p})(0).$$

Theorem 3.4.2 implies that the  $\Theta$ -lift of the parameter for  $\mathcal{O}_1$  corresponding to  $(\eta_1, \dots, \eta_p)$  goes to the parameter  $(\eta_0 = -1, \eta_1, \dots, \eta_p)$  for  $\mathcal{O}_0$ . The parameters with  $\eta_0 = 1$  are  $\Theta$ -lifts of the parameters of  $\mathcal{O}_1$  with  $K$ -types with a  $+$ .

For  $\mathcal{O}_1$  of type  $D$  to  $\mathcal{O}_0$  of type  $C$ , an even column  $c_0$  larger than  $m_0$  is added, and  $m_0 \rightarrow c_1, \dots, m_{2p+1} \rightarrow c_{2p+2}$ . The columns of the ensuing  $\mathcal{O}_0$  are paired

$$(c_0, m_0)(m_1, m_2) \dots (m_{2p-1}, m_{2p})(m_{2p+1})$$

Theorem 3.4.2 implies that the  $\Theta$ -lift of the parameter for  $\mathcal{O}_1$  corresponding to  $(\eta_1, \dots, \eta_p)$  goes to the parameter  $(\eta_0 = -1, \eta_1, \dots, \eta_p)$  for  $\mathcal{O}_0$ . The parameters with  $\eta_0 = 1$  are  $\Theta$ -lifts of the parameters of  $\mathcal{O}_1$  with  $K$ -types with a  $+$ .

□

We abbreviate  $Sp(2n), O(m)$  for  $Sp(2n, \mathbb{C}), O(m, \mathbb{C})$  and similarly for the Lie algebras.

**Example 3.5.2.** Consider the nilpotent orbit in  $so(8)$  with columns  $\mathcal{O} \longleftrightarrow (4, 3, 1)$ . The infinitesimal character is  $(1, 0, 3/2, 1/2)$ . Then  $(V_0, 1)$  is of dimension 8, and  $(V_1, -1)$  is of dimension 4.  $\mathcal{O}_1 \longleftrightarrow (3, 1)$  and the unipotent representations are the two oscillator representations

$$\begin{pmatrix} 3/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 3/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}$$

They correspond to the two unipotent representations of  $SO(8)$  with parameters

$$\begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 1 & 0 & 3/2 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 1 & 0 & 3/2 & -1/2 \end{pmatrix}$$

**Example 3.5.3.** Let  $\mathcal{O}_1 \longleftrightarrow (4, 2, 2)$  in  $so(8)$ . It matches  $\mathcal{O}_0 \longleftrightarrow (4, 4, 2, 2)$  in  $sp(12)$ . The infinitesimal characters are  $(1, 1, 0, 0)$  and  $(2, 1, 1, 1, 0, 0)$ . The parameters for  $\mathcal{O}_1$  are

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The parameters for  $\mathcal{O}$  are

$$\begin{pmatrix} 2 & 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 & -1 & 0 & -1 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 & -2 & 0 & -1 \\ 2 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

The second column is obtained by applying the correspondence to the parameters for  $O(8)$  tensored with  $\text{sgn}$ .

**Example 3.5.4.** Let  $\mathcal{O} \longleftrightarrow (33)$  in  $sp(6)$ . Then  $\mathcal{O}_1 \longleftrightarrow (3)$  in  $so(3)$ . The infinitesimal characters are  $(3/2, 1/2, 1/2)$  and  $(1/2)$  as given by the previous algorithms.

The rows of  $\mathcal{O}$  are  $(2, 2, 2)$  there is only one special unipotent representation, its infinitesimal character is  $(1, 1, 0)$ . By contrast infinitesimal character  $(3/2, 1/2, 1/2)$  matches the  $\Theta$ -correspondence and there are two parameters.

**Example 3.5.5.** Let  $\mathcal{O} \longleftrightarrow (4, 2, 2)$  in  $sp(8)$ . It corresponds to  $\mathcal{O} \longleftrightarrow (2, 2)$  in  $so(4)$ . There are two such nilpotent orbits if we use  $SO(4)$ , one if we use  $O(4)$ . We will use orbits of the orthogonal group. The infinitesimal character corresponding to  $(2, 2)$  is  $(1/2, 1/2)$ . The representations corresponding to  $(4, 2, 2)$  have infinitesimal character  $(1, 0, 1/2, 1/2)$ . The Langlands parameters are spherical

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 1 & 0 & 1/2 & 1/2 \end{pmatrix}$$

We can go further and match  $(2, 2)$  in  $so(8)$  with  $(2)$  in  $sp(2)$ . If we combine these steps we get infinitesimal characters  $(1) \mapsto (0, 1) \mapsto (2, 1, 0, 1)$ .

There is nothing wrong with the correspondence of irreducible modules. But note that the infinitesimal character  $(2, 1, 1, 0)$  has maximal primitive ideal corresponding to the orbit  $\mathcal{O} \longleftrightarrow (4, 4)$ , (rows  $(2, 2, 2, 2)$ ).

This is one of the reasons for imposing the conditions on the nilpotent orbits, we want to be able to iterate and stay within the class of unipotent representations. One obtains induced modules with interesting composition series. In this example, let  $P$  be the parabolic subgroup with Levi component  $GL(2) \times Sp(4)$  and  $\chi$  be a character on  $GL(2)$  so that the induced module  $\text{Ind}_P^{Sp(8)}[\chi \otimes \text{Triv}]$  has infinitesimal character  $\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}}$  with  $\lambda_{\mathcal{O}} = (2, 1, 1, 0)$ . Then

$$\text{Ind}_{GL(2) \times Sp(4)}^{Sp(8)}[\chi \otimes \text{Triv}] = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 2 & 1 & 2 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

The first two parameters are unipotent, corresponding to  $\mathcal{O} \longleftrightarrow (4, 4)$ . The last factor is bigger, the annihilator corresponds to the nilpotent orbit  $(4, 2, 2)$ . All these composition factors have nice character formulas analogous to those for the special unipotent representations even though their annihilators are no longer maximal. Daniel Wong has made an extensive study of these representations in his thesis.

This example is tied up with the fact that nilpotent orbits are not always normal. A nilpotent orbit is normal if and only if  $R(\mathcal{O}) = R(\overline{\mathcal{O}})$ . The orbit  $(4, 2, 2)$  is **not normal**.  $R(\mathcal{O})$  is the full induced representation from a 1-dimensional representation of  $\mathfrak{m} = \mathfrak{gl}(2) \times \mathfrak{sp}(4)$ .  $R(\overline{\mathcal{O}})$  is the sum of the first and last representation, missing the middle one. These equalities are in the sense that the  $K$ -types of the representations match the  $G$ -types of the regular functions, using the identification  $K_{\mathbb{C}} \cong G$ . It is **not** the case that  $R(\mathcal{O})$  and  $R(\overline{\mathcal{O}})$  are representations of  $G$  as a real Lie group.  $\square$

#### 4. REGULAR FUNCTIONS ON NILPOTENT ORBITS AND UNIPOTENT REPRESENTATIONS

**4.1. Background Results.** Most of the details in this section can be found in [McG], [G], and references therein.

The structure sheaf of a variety  $Z$  will be denoted by  $\mathcal{S}_Z$ . We will abbreviate  $R(Z)$  for  $\Gamma(Z, \mathcal{S}_Z)$ .

Typically  $\mathcal{O}$  will denote the orbit of a nilpotent element  $e$  in a reductive Lie algebra  $\mathfrak{g}$ . The orbit is isomorphic to  $G/G(e)$ . Its universal cover  $\tilde{\mathcal{O}}$  is isomorphic to  $G/G(e)_0$ . By one of Chevalley's theorems there is a representation  $\tilde{V}$  and a vector  $\tilde{e} = (e, \tilde{v}) \in \mathfrak{g} \oplus \tilde{V}$  such that its orbit under  $G$  is the universal cover; in other words the stabilizer of  $\tilde{v}$  is  $G(e)_0$ . Given any closed subgroup  $G(e)_0 \subset H \subset G(e)$ , there is a corresponding cover  $\tilde{\mathcal{O}}_H$  which can be realized as the orbit of  $G$  of an element  $e_H = (e, v_H) \in \mathfrak{g} \oplus V_H$ .



4.2. Let  $\{e, h, f\}$  be a Lie triple associated to  $e$ . Let  $\mathfrak{g}_{\geq 2}$  be the sum of the eigenvectors of  $\text{ad } h$  with eigenvalue greater than or equal to 2. Let  $P(e)$  be the parabolic subgroup determined by  $h$ , *i.e.* the parabolic subgroup corresponding to the roots with eigenvalue greater than or equal to zero for  $\text{ad } h$ . It is well known that the natural map

$$(16) \quad m_e : G \times_{P(e)} \mathfrak{g}_{\geq 2} \longrightarrow \overline{\mathcal{O}}, \quad (g, X) \mapsto gXg^{-1}$$

is birational and projective. The birationality follows from [BV]. The projective property is in [McG].

4.3. The notions and results in the next sections are for  $G$  the rational points of a reductive group over an algebraically closed field of characteristic 0.

Let  $P = MN$  be an arbitrary parabolic subgroup and  $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$  be a nilpotent orbit. A  $G$ -orbit  $\mathcal{O}$  is called *induced* from  $\mathcal{O}_{\mathfrak{m}}$  ([LS]), if

$$(17) \quad \mathcal{O} \cap [\mathcal{O}_{\mathfrak{m}} + \mathfrak{n}] \text{ is dense in } \mathcal{O}_{\mathfrak{m}} + \mathfrak{n}.$$

Let  $\Sigma := \mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$ . There is a similar *moment map*

$$(18) \quad m : G \times_P \Sigma \longrightarrow \overline{\mathcal{O}}, \quad (g, X) \mapsto gXg^{-1}.$$

It is projective for the same reason as before, but it is not always birational. Precisely, if  $e \in \Sigma \cap \mathcal{O}$ , then the generic fiber of  $m$  is isomorphic to  $G(e)/P(e)$ .

We will write  $\mathcal{Z}$  for  $G \times_P \Sigma$  where  $\Sigma = \mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$ . In general, write  $A_G(e) := G(e)/G(e)_0$ . We suppress the subscript  $G$  if it is clear from the context. Recall from [LS] that  $G(e)_0 = P(e)_0$ , so that there is an inclusion  $A_P(e) \subset A_G(e)$ . If  $e_{\mathfrak{m}} \in \mathcal{O}_{\mathfrak{m}}$ , then there is a surjection  $A_P(e) \rightarrow A_M(e_{\mathfrak{m}})$ . Given a representation  $\phi$  of  $A_M(e_{\mathfrak{m}})$ , we will denote by the same letter  $\phi$  its inflation to  $A_P(e)$ .

4.4. Given a (cover of) an orbit  $\mathcal{O} \cong G/G(e)$ , recall from [J] section 8.1 that

$$R(\mathcal{O}) = \text{Ind}_{G(e)}^G [\text{Triv}] \quad (\text{algebraic induction}).$$

**Definition 4.4.1.** Let  $\Psi \in \widehat{G(e)}$  be trivial on  $G(e)_0$ , and write

$$R(\mathcal{O})_{\Psi} = \text{Ind}_{G(e)}^G [\Psi].$$

Regular functions on the universal cover  $\tilde{\mathcal{O}}$  satisfy

$$R(\tilde{\mathcal{O}}) = \sum_{\Psi \in \widehat{G(e)_0}} \text{Ind}_{G(e)}^G \text{Ind}_{G(e)_0}^{G(e)} [\Psi].$$

4.5. Let  $e_{\mathfrak{m}} \in \mathcal{O}_{\mathfrak{m}}$  and  $\lambda \in \mathfrak{m}$  be semisimple such that  $C_{\mathfrak{g}}(\lambda) = \mathfrak{m}$ , and  $\mathfrak{n}$  is spanned by the root vectors of roots positive on  $\lambda$ . Let  $e \in e_{\mathfrak{m}} + \mathfrak{n}$  be a representative for the induced nilpotent. Let  $\psi$  be a representation of  $A_M(e_{\mathfrak{m}})$  (equivalent to the inflated representation on  $A_P(e)$ ) and  $\Psi$  be the induced representation to  $A_G(e)$ . Choose a ( $K$ -invariant) inner product on  $\mathfrak{g}$ . By Frobenius reciprocity,

$$R(\mathcal{O})_{\Psi} := \sum_{\rho \in \widehat{A(\mathcal{O})}} [\rho|_{A_M(e_{\mathfrak{m}})} : \psi] R(\mathcal{O})_{\rho}.$$

**Proposition 4.5.1.** *Let  $(\mu, V)$  be a representation of  $G$ . Then*

$$[\mu : \text{Ind}_M^G[R(\mathcal{O}_{\mathfrak{m}})_{\psi}]] \leq [\mu : R(\mathcal{O})_{\Psi}].$$

*Proof.* We work with  $G(e)$  and  $M(e_{\mathfrak{m}})$  with  $\psi$  and  $\psi$  trivial on the connected component of the identity so in particular also trivial on the corresponding unipotent radicals. For  $n \in \mathbb{N}$ , consider  $\frac{1}{n}\lambda + e_{\mathfrak{m}}$ . There is  $p_n \in K \cap P$  such that  $\lambda_n := \text{Ad}(p_n)(\frac{1}{n}\lambda + e_{\mathfrak{m}}) = \frac{1}{n}\lambda + e$ . The centralizer of  $\lambda - n$  has the same dimension as the centralizer of  $e$ . We show that for every  $(\mu, V)$  linearly independent vectors transforming under  $M(e_{\mathfrak{m}})$  according to  $\psi$  give rise to vectors transforming according to  $\Psi$  under  $G(e)$ .

For each  $n$ , let  $X_n^1, \dots, X_n^k$  be an orthonormal basis of  $C_{\mathfrak{g}}(\lambda_n)$ , the centralizer in  $\mathfrak{g}$  of  $\lambda_n$ . We can extract a subsequence such that the  $X_n^i$  all converge to an orthonormal basis of  $C_{\mathfrak{g}}(e)$ . Now let  $v_1^n, \dots, v_l^n$  be an orthonormal basis of the space of fixed vectors of  $C_{\mathfrak{g}}(\lambda_n)$  in  $V$ . We can again extract a subsequence such that the  $v_n^j$  all converge to an orthonormal set of vectors in  $V$ . Because  $v_n^j$  are invariant under the action of the  $X_n^i$ , their limits are invariant under an orthonormal basis of  $C_{\mathfrak{g}}(e)$ . Using Frobenius reciprocity, this proves the claim for the connected components of the centralizers, *i.e.* the corresponding statement for  $R(\tilde{\mathcal{O}}_{\mathfrak{m}})$  and  $R(\tilde{\mathcal{O}})$ . The proof of the general case is a straightforward modification, using the fact that  $A_G(e)$  and  $A_M(e_{\mathfrak{m}})$  are finite groups.  $\square$

4.6. For the case of a Richardson nilpotent orbit, the previous result can be sharpened as follows. The details are in [J] chapter 8. Let  $P = MN$  be a parabolic subgroup with Lie algebra  $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ . Denote again by  $\lambda \in \mathfrak{g}$  a semisimple element whose centralizer is  $\mathfrak{m}$ , and which is positive on the roots of  $\mathfrak{n}$ . Let  $e \in \mathfrak{n}$  be a representative of the Richardson induced orbit from this parabolic subalgebra, and denote its  $G$  orbit by  $\mathcal{O}$ . As before, there is a map

$$(19) \quad m : \mathcal{P}_{\mathfrak{n}} := G \times_P \mathfrak{n} \longrightarrow \mathfrak{g},$$

with image  $\overline{\mathcal{O}}$ . Let  $\tilde{\mathcal{O}}$  be the inverse image of  $\mathcal{O}$ . By lemma 8.8 in [J],  $\tilde{\mathcal{O}}$  is a single  $G$ -orbit, and an open dense subset of  $\mathcal{P}_{\mathfrak{n}}$ . In addition  $\tilde{\mathcal{O}}$  is an unramified cover of  $\mathcal{O}$  with fiber  $A_G(\mathcal{O})/A_P(\mathcal{O})$ . Identify representations of  $A_P(e)$  and  $A_G(e)$  with representations of  $G(e)$  by making them trivial on  $G(e)_0$ .

**Proposition 4.6.1.**

$$[\mu : \text{Ind}_P^G[\text{triv}]] = \sum_{\rho \in \widehat{A_G(e)}} [\rho|_{A_P(e)} : \text{triv}][\mu : R(\mathcal{O})_\rho].$$

*Proof.* By formula (4) in section 8.9 of [J],

$$R[\tilde{\mathcal{O}}] \cong R[\mathcal{P}_{\mathfrak{n}}] \cong \bigoplus_n H^0[G/P, S^n(\mathfrak{n}^*)],$$

where  $\mathcal{P}_{\mathfrak{n}} := G \times_P \mathfrak{n}$ . Theorem 8.15 in [J] says that

$$H^i[G/P, S^n(\mathfrak{n}^*)] = 0 \quad \text{for all } i > 0, n \in \mathbb{N}.$$

The final formula follows by the standard relations between  $H^i(G/P, V)$  and  $\mathfrak{n}$ -cohomology.  $\square$

We will use this proposition in the setting of a triangular nilpotent orbit in a classical type Lie algebra, and  $P$  such that  $A_P(e) = \{1\}$ .

4.7. We return to the case where  $P$  corresponds to the middle element of the Lie triple. Write  $A(\mathcal{O}) := A_G(e)$ . In this case,  $G(e) \subset P$ . Recall  $\Sigma := \mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$  and  $\mathcal{Z} := G \times_P \Sigma$ . Let  $\chi \in \widehat{A(\mathcal{O})}$  be an irreducible representation viewed as a representation of  $G(e)$  trivial on  $G(e)_0$ , and assume there is a representation  $\xi$  of  $P$  such that  $\xi|_{G(e)} = \chi$ . Then

$$(20) \quad H^0(G/P, R(P \cdot e) \otimes \mathbb{C}_\xi) \subset R(\mathcal{O}, \mathcal{S}_\chi)$$

because  $\mathcal{O}$  embeds in  $\mathcal{Z}$  via  $g \cdot e \mapsto [g, e]$ . The results in [McG] imply that there is equality. Indeed, if  $\phi \in R(\mathcal{O}, \mathcal{S}_\chi)$ , view it as a map  $\phi : G \rightarrow \mathbb{C}$  satisfying

$$(21) \quad \phi(gx) = \chi(x^{-1})\phi(g).$$

Then define a section  $s_\phi \in H^0(G/P, R(P \cdot e))$  by the formula

$$(22) \quad s_{\phi, \xi}(g)(p \cdot e) := \xi(p)\phi(gp).$$

The inverse map is given by

$$(23) \quad s \mapsto \phi_s(g) := s(g)(e).$$

There is another inclusion

$$(24) \quad H^0(G/P, R(\mathfrak{g}_{\geq 2} \otimes \mathbb{C}_\chi) \subset H^0(G/P, R(P \cdot e) \otimes \mathbb{C}_\xi).$$

In [McG] it is shown that when  $\chi = \text{triv}$  and  $\xi = \text{triv}$ , then equality holds in (24), and in addition

$$(25) \quad H^i(G/P, R(\mathfrak{g}_{\geq 2})) = (0) \quad \text{for } i > 0.$$

These results suggest the following conjecture.

**Conjecture 4.7.1.** *For each  $\chi \in \widehat{A(\mathcal{O})}$  there is a representation  $\xi$  of  $P(e)$  satisfying  $\xi|_{G(e)} = \chi$  such that*

$$H^i(G/P(e), R(\mathfrak{g}_{\geq 2}) \otimes \mathcal{S}_\xi) = \begin{cases} R(\mathcal{O})_\chi, & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

4.8. Recall Lusztig's quotient of the component group  $A(\mathcal{O})$  denoted  $\overline{A(\mathcal{O})}$ , and the definition of cuspidal and stably trivial orbits.

**Definition 4.8.1.** *An orbit is called cuspidal if it is not induced from any nilpotent orbit in a proper Levi component. A more common terminology is rigid.*

*A special orbit satisfying  $A(\mathcal{O}) = \overline{A(\mathcal{O})}$  is called smoothly cuspidal.*

Smoothly cuspidal orbits have the property that the dual orbit  $\check{\mathcal{O}}$  is even. They are listed below, and not necessarily cuspidal.

Let  $\check{e}, \check{h}, \check{f}$  be a Lie triple associated to  $\check{\mathcal{O}}$ . In these cases  $\lambda_{\mathcal{O}} = \check{h}/2$ . So these are the parameters treated in [BV2], also referred to as *special unipotent*.

Precisely, in terms of partitions, smoothly cuspidal orbits for classical groups are as follows.

- (B): Every row size except the largest one occurs an even number of times. Also the columns are  $(m_0)(m_1, m_2) \dots (m_{2p-1}, m_{2p})$  with  $m_{2k} > m_{2k+1}$ , and all columns have odd size.
- (C): Every row size occurs an even number of times. Also the columns are  $(c_0, c_1) \dots (c_{2p-2}, c_{2p-1})(c_{2p})$  with  $c_{2j-1} > c_{2j}$ , and all column sizes are even.
- (D): Every row size occurs an even number of times. Also the columns are  $(m_0, m_{2p+1})(m_1, m_2) \dots (m_{2p-1}, m_{2p})$  with  $m_{2j} > m_{2j+1}$ , and all column sizes are even.

4.9. View the complex group  $G$  as a real Lie group, and let  $K$  be the maximal compact subgroup. Then  $R(\mathcal{O})$  can be thought of as a  $K$ -module using the identification of  $K_c$  with  $G$ .

Given  $\chi \in \widehat{A(\mathcal{O})}$ , denote by  $R(\mathcal{O})_\chi$  the regular sections of the sheaf corresponding to  $\chi$ . We summarize the statements in the paper in the following conjecture, which can be thought of as a sharpening of the material in Section 2.3. Recall the notion of *Associated Variety* from [V]. A review of these notions and relations to the *Associated Cycle* is in later sections.

We identify nilpotent  $G$ -orbits in the real algebra  $\mathfrak{g}$  with  $K_c$ -orbits in  $\mathfrak{p}_c$  via the Kostant-Sekiguchi correspondence. Using the identification  $\mathfrak{g}_c = \mathfrak{g} \times \mathfrak{g}$  and  $K_c \cong G$ ,  $K_c$ -orbits in  $\mathfrak{p}_c$  are identified with  $G$ -orbits in  $\mathfrak{g}$ , this time considered as a complex group and complex vector space respectively.

**Conjecture 4.9.1.** *Given a nilpotent orbit  $\mathcal{O}$ , there is an infinitesimal character  $\lambda_{\mathcal{O}}$  with the following property.*

*There is a 1-1 correspondence  $\chi \longleftrightarrow X_{\chi}$  between characters of the component group and irreducible  $(\mathfrak{g}, K)$  modules with  $\mathcal{O}$  as associated cycle and infinitesimal character  $\lambda_{\mathcal{O}}$  with the following properties:*

- (1) *The analogous character formulas as in [BV2] hold,*
- (2)  *$X_{\chi}$  are unitary,*
- (3)  *$X_{\chi}|_K \cong R(\mathcal{O})_{\chi}$*

The  $\lambda_{\mathcal{O}}$  given in section 2.3 satisfy (2) by the unitarity results in [B1]; the character formulas in (1) are generalizations of those in [BV2] using the Kazhdan-Lusztig conjectures for nonintegral infinitesimal character (also in [B1]).

4.10. Theorem 4.10.1 below provides evidence for (3).

**Theorem 4.10.1.** *Assume  $\mathcal{O}$  is smoothly cuspidal, and  $\mathfrak{g}$  is of classical type. There is a correspondence  $\chi \longleftrightarrow X_{\chi}$  between characters of  $A(\mathcal{O})$  and unipotent representations determined by the property*

$$X_{\chi}|_K \cong R(\mathcal{O})_{\chi}.$$

By the results in [B1], the representations  $X_{\chi}$  are unitary as well. The proof will be given in Section 5.2.

4.11. For the spherical case, theorem 4.10.1 is more general.

**Theorem 4.11.1.** *Assume  $\mathcal{O}$  is arbitrary,  $\mathfrak{g}$  is of classical type, and let  $L_{triv}$  be the spherical module with infinitesimal character  $\chi$  defined in sections 2.3 to 2.8. Then*

$$R(\mathcal{O}) \cong L_{triv}|_K$$

These theorems imply that for the case of a complex classical group,  $R(\mathcal{O})_{\chi}$  is realized as the  $K$ -spectrum of a  $(\mathfrak{g}, K)$ -module. In particular,  $R(\mathcal{O})$  can be written as a combination of standard modules with the same infinitesimal character, not just as a combination of tempered modules as in [McG].

**4.12. Associated Cycle of an Admissible Module.** We review the results in [V1] and [V2] which will be crucial for the proof of the above theorems. Denote by  $\mathcal{M}(\mathfrak{g}, K)$  the category of admissible  $(\mathfrak{g}, K)$ -modules.

Recall  $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$  the complexification of the Cartan decomposition of a real reductive algebra  $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ . Let  $(\pi, X)$  be an admissible  $(\mathfrak{g}, K)$ -module. Section 2 of [V1], attaches to  $(\pi, X)$  a  $(S(\mathfrak{g}), K)$  module  $(gr(\pi), gr(X))$ . This module is finitely generated, graded. Attached to any  $S(\mathfrak{g})$ -module  $M$  (equal to  $gr(X)$ ) are varieties

$$(26) \quad \mathcal{V}(M) \supset Supp(M) \supset Ass(M),$$

the set of prime ideals containing the annihilator of  $gr(M)$ , the support of  $gr(M)$ , and the set of *associated primes*, those primes in  $\mathcal{V}(M)$  which are annihilators of elements in  $gr(M)$ . Since  $\mathfrak{k}$  acts by zero,  $gr(X)$  is in fact an  $S(\mathfrak{g}/\mathfrak{k}) \cong S(\mathfrak{s})$ -module. So the sets in (26) are all  $K_{\mathbb{C}}$ -invariant varieties in  $\mathfrak{s}$ . Since the module  $X$  was assumed admissible,  $M = gr(X)$  is finitely generated, so  $\mathcal{V}(M) = Supp(M)$ , and  $Ass(M)$  is finite containing the minimal primes of  $\mathcal{V}(M)$ . In particular the varieties corresponding to  $Ass(M)$  and  $\mathcal{V}(M)$  coincide. The center of  $U(\mathfrak{g})$  must act by generalized eigenvalues on an admissible module, so  $S(\mathfrak{g})^{\mathfrak{g}}$  acts by 0 on  $M$ . Thus the sets in (26) are contained in  $\mathcal{N}_{\theta} := \mathcal{N} \cap \mathfrak{s}$ . We will write  $\mathcal{V}(X)$ ,  $Supp(X)$ , and  $Ass(X)$  for the corresponding objects for  $M = gr(X)$ .

Denote by  $\mathcal{C}(\mathfrak{g}, K)$  (definition 6.8 in [V2]) the category of finitely generated  $S(\mathfrak{g}/\mathfrak{k})$ -modules  $N$  carrying locally finite representations of  $K$ , subject to

$$(27) \quad \begin{aligned} k \cdot (p \cdot n) &= (\text{Ad}(k)p) \cdot (k \cdot n) & k \in K_{\mathbb{C}}, p \in S(\mathfrak{g}), n \in N, \\ \mathcal{V}(N) &\subset \mathcal{N}_{\theta}. \end{aligned}$$

Proposition 2.2 in [V1] states that the map  $gr$  gives rise to a well defined map  $Kgr$  between the Grothendieck groups  $K\mathcal{M}(\mathfrak{g}, K)$  to  $K\mathcal{C}(\mathfrak{g}, K)$ . Furthermore  $X$  and  $M = gr(X)$  have the same  $K$ -structure. Choose representatives  $\lambda_1, \dots, \lambda_r$  for the nilpotent  $K_{\mathbb{C}}$ -orbits, and let  $H_i$  be the corresponding isotropy subgroups. The support of any nonzero module  $N \in \mathcal{C}(\mathfrak{g}, K)$  can be written uniquely as a union of closures  $K_{\mathbb{C}} \cdot \lambda_i$  where  $\lambda_i$  is not in the cloure of any other orbit in the support. Following (7.4)(b) and (7.4)(c) of [V2], let

$$(28) \quad \begin{aligned} \mathcal{C}(\mathfrak{g}, K)_i &:= \{N \in \mathcal{C}(\mathfrak{g}, K) \mid \lambda_i \in \overline{(K_{\mathbb{C}} \cdot \lambda_j)} \setminus K_{\mathbb{C}} \cdot \lambda_j \Rightarrow \lambda_j \notin \mathcal{V}(N)\}, \\ \mathcal{C}(\mathfrak{g}, K)_i^0 &:= \{N \in \mathcal{C}(\mathfrak{g}, K) \mid \lambda_i \notin \mathcal{V}(N)\}. \end{aligned}$$

**Theorem 4.12.1** (2.13 in [V1], proposition 7.6 in [V2]). *Attached to any  $N \in \mathcal{C}(\mathfrak{g}, K)$  there is a genuine virtual representation  $\chi(\lambda_i, N)$  of  $H_i$  with the following property.*

*This correspondence descends to an isomorphism of Grothendieck groups*

$$K\mathcal{C}(\mathfrak{g}, K)_i / K\mathcal{C}(\mathfrak{g}, K)_i^0 \cong K\mathcal{F}(H_i)$$

*where  $K\mathcal{F}(H_i)$  is the Grothendieck group of (algebraic) representations of  $H_i$ .*

**Proposition 4.12.2** (proposition 7.9 in [V2]). *Suppose that  $(\tau, V_{\tau})$  is an irreducible representation of  $H_i$ . There is an object  $N(\lambda_i, \tau) \in \mathcal{C}(\mathfrak{g}, K)$  such that:*

- (1)  $\mathcal{V}((N, \lambda_i, \tau)) = \overline{K_{\mathbb{C}} \cdot \lambda_i}$ .
- (2)  $\chi(N(\lambda_i, \tau)) = \tau$ .

*Any such choice of  $\{N(\lambda_i, \tau)\}$  gives rise to a basis  $[N(\lambda_i, \tau)]$  of  $K\mathcal{C}(\mathfrak{g}, K)$ .*

*When  $\overline{K_{\mathbb{C}} \cdot \lambda_i}$  has no orbits of codimension 1 in its closure, one can choose*

$$N(\lambda_i, \tau) = \text{Ind}_{H_i}^K \tau.$$

**Corollary 4.12.3** (4.11 and 4.7 in [V1], and [V2]). *Assume that  $G$  is the real points of a complex reductive group. A basis of  $K\mathcal{C}(\mathfrak{g}, K)$  is formed of*

$$\left\{ \text{Ind}_{H_i}^K \tau \right\}_{i=1, \dots, r, \tau \in \widehat{H_i}}.$$

*The support of any irreducible  $(\mathfrak{g}, K)$ -module  $X$  is the closure of a single orbit  $\mathcal{O}$ . Furthermore,*

$$X|_K = \text{Ind}_{H_i}^K \chi(\text{gr}(X), \mathcal{O}) - D(X)$$

*where  $D(X) \in \mathcal{C}(\mathfrak{g}, K)$  with support strictly smaller than  $\mathcal{O}$ .*

**Definition 4.12.4.** *The **associated cycle**  $AC(X)$  of an admissible  $(\mathfrak{g}, K)$ -module  $X$  is the formal sum*

$$AC(X) := \sum (\dim \chi_i) \mathcal{O}_i$$

*where  $\mathcal{V}(\text{gr}(X)) = \cup \overline{\mathcal{O}_i}$ , are the irreducible components, and  $\chi_i = \chi(\text{gr}(X), \mathcal{O}_i)$ .  $\dim \chi_i$  is called the multiplicity of  $\mathcal{O}_i$  in the associated cycle of  $X$ .*

**4.13. Asymptotic Cycle for Induced Modules.** We follow [BV1] section 3. Let

$$\Omega := \{X \in \mathfrak{g} : |\text{Im} \lambda| < \pi, \text{ for any eigenvalue } \lambda \text{ of } \text{ad } X\}.$$

Then  $\Omega$  is invariant under  $\text{Ad } G$ , and there is an open neighborhood  $\mathcal{V}$ , of the identity  $e \in G$  such that  $\exp : \Omega \rightarrow \mathcal{V}$  is an isomorphism.

Next define functions

$$j(X) = \det \left[ \frac{e^{\text{ad } X/2} - e^{-\text{ad } X/2}}{\text{ad } X} \right],$$

$$\xi(X) := j(X)^{1/2}, \quad \xi(0) = 1.$$

The Haar measure  $dx$  on  $G$  is related to Lesbegue measure on  $\mathfrak{g}$  by  $dx = \xi(X)^2 dX$ . There is a map

$$(29) \quad \begin{aligned} \phi &\in C_c^\infty(\Omega) \mapsto f_\phi \in C_c^\infty(G) \\ f_\phi(\exp X) &:= \xi(X)^{-1} \phi(X). \end{aligned}$$

This induces a map on the level of distributions

$$(30) \quad \begin{aligned} \Theta &\in \mathcal{D}(\mathcal{V}) \mapsto \theta \in \mathcal{D}(\Omega), \\ \theta(\phi) &:= \Theta(f_\phi). \end{aligned}$$

This map takes  $G$ -invariant eigendistributions of the center of the enveloping algebra  $U(\mathfrak{g})$  to invariant eigendistributions on  $\Omega$  of the constant coefficient  $G$ -invariant operators  $\partial(I(\mathfrak{g}))$  on  $\mathfrak{g}_\mathbb{C}$ .

Let  $P = MAN$  be a parabolic subgroup, and  $(\pi, \mathcal{H})$  an admissible  $(\mathfrak{m}, M \cap K)$ -module. For  $\nu \in \mathfrak{a}_\mathbb{C}^*$ , where  $\mathfrak{a} := \text{Lie}(A)$ , let  $\pi_P$  be the module equal to  $\pi_P(man) := e^{(\nu - \rho)(\log a)} \pi(m)$ , where  $\rho := \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})} \alpha$ . Let  $\pi_\nu$  be the induced module.

**Lemma 4.13.1** (Lemma 3.3 in [BV1]). *Let  $\Theta := \text{Tr}(\pi)$  and  $\Theta_\nu := \text{Tr}(\pi_\nu)$ . Let  $\phi \in C_c^\infty(P)$  and  $f \in C_c^\infty(G)$ . Then*

$$\begin{aligned} \text{Tr } \pi_P(\phi) &= \int_{MAN} e^{(\rho+\nu)(\log a)} \Theta(m) \phi(man) \, dm \, da \, dn, \\ \Theta_\nu(f) &= \int_{MA} e^{(\rho+\nu)(\log a)} \Theta(m) \int_{KN} f(kmank^{-1}) \, dk \, dn \, dm \, da = \\ &= \int_{MA} e^{\nu(\log a)} \Theta(m) D(ma) \int_{G/MA} f(xmax^{-1}) \, dx \, dm \, da, \end{aligned}$$

where  $ma = \exp(X_{\mathfrak{m}} + X_{\mathfrak{a}})$  and

$$D(\exp(X_{\mathfrak{m}} + X_{\mathfrak{a}})) = |\det(e^{\text{ad}(X_{\mathfrak{m}}+X_{\mathfrak{a}})} - e^{-\text{ad}(X_{\mathfrak{m}}+X_{\mathfrak{a}})})|_{\mathfrak{n}}|.$$

Let  $\theta$  and  $\theta_\nu$  be the lifts of  $\Theta$  and  $\Theta_\nu$ . Plug in  $f = f_\phi$ :

$$\begin{aligned} (31) \quad \theta_\nu(\phi) &= \int_{\mathfrak{m}+\mathfrak{a}} e^{(\rho+\nu)(X_{\mathfrak{a}})} \Theta(\exp X_{\mathfrak{m}}) \xi(X_{\mathfrak{m}} + X_{\mathfrak{a}})^{-1} \cdot \\ &\quad \cdot \int_{K \times \mathfrak{n}} \phi(\text{Ad } k(X_{\mathfrak{m}} + X_{\mathfrak{a}} + X_{\mathfrak{n}})) \, dk \, dX_{\mathfrak{n}} \, dX_{\mathfrak{m}} \, dX_{\mathfrak{a}}. \end{aligned}$$

Decompose  $\xi = \xi_{MA} \cdot \xi_N$ , and denote by

$$(32) \quad \phi_P(X_{\mathfrak{m}} + X_{\mathfrak{a}}) = \int_{\mathfrak{n}} \int_K \phi(\text{Ad } k(X_{\mathfrak{m}} + X_{\mathfrak{a}} + X_{\mathfrak{n}})) \, dX_{\mathfrak{n}} \, dk.$$

Formula (31) becomes

$$(33) \quad \theta_\nu(\phi) = \int_{\mathfrak{m}+\mathfrak{a}} e^{(\rho+\nu)(X_{\mathfrak{a}})} \theta(\exp X_{\mathfrak{m}}) \xi_N(X_{\mathfrak{m}} + X_{\mathfrak{a}})^{-1} \cdot \phi_P(X_{\mathfrak{m}} + X_{\mathfrak{a}}) \, dX_{\mathfrak{m}} \, dX_{\mathfrak{a}}.$$

Recall from [BV1],  $\phi_t(X) := t^{\dim \mathfrak{g}} \phi(t^{-1}X)$ . Then

$$(\phi_t)_P = t^{-\dim \mathfrak{n}} (\phi_P)_t.$$

It follows that if the asymptotic expansion of  $\Theta$  has leading term  $D_r$ , then the leading term of the asymptotic expansion of  $\Theta_\nu$  is  $D_r(\phi_P)$ , but at degree  $r + \dim \mathfrak{n}$ .

Write  $\mathfrak{g} = \bar{\mathfrak{n}} + (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n}$ . Denote by  $\mathcal{F}_{\mathfrak{g}}$  and  $\mathcal{F}_{\mathfrak{m}+\mathfrak{a}}$  the Fourier transforms with respect to the Cartan-Killing form of  $\mathfrak{g}$  and the Cartan-Killing form of  $\mathfrak{g}$  restricted to  $\mathfrak{m} + \mathfrak{a}$  respectively. Formula (32) defines a map  $\phi \in C_c^\infty(\mathfrak{g}) \rightarrow \phi_P \in C_c^\infty(\mathfrak{m} + \mathfrak{a})$ .

**Lemma 4.13.2.**

$$\mathcal{F}_{\mathfrak{g}}(\phi)_{\overline{P}} = \mathcal{F}_{\mathfrak{m}+\mathfrak{a}}(\phi_P)$$

Recall from [BV1] that the leading term of a character  $AS(\Theta)$  is a combination of Fourier transforms of Liouville measures of nilpotent orbits,  $AS(\Theta) = \sum c_j \widehat{\mu(\mathcal{O}_j)}$ . We call  $AS(\Theta)$  the **asymptotic cycle** of  $\Theta$ .



**Definition 4.13.3.** Let  $D$  be a tempered  $MA$ -invariant homogeneous distribution. Denote by  $\text{Ind}_P^G D$  the distribution

$$\text{Ind}_P^G[D](\phi) := D(\phi_P).$$

When  $D$  is the invariant measure of a nilpotent orbit  $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$ ,  $\text{Ind}_P^G D$  is a combination of invariant measures supported on nilpotent orbits of  $\mathfrak{g}$ .

**Corollary 4.13.4.** Using the notation  $\theta$  for the character of  $(\pi_M, \mathcal{H})$  and  $\theta_\nu$  for the induced character, suppose  $AS(\theta) = \sum c_j \mathcal{F}_{\mathfrak{m}+\mathfrak{a}}(\mu(\mathcal{O}_{j,\mathfrak{m}}))$ . Then  $AS(\theta_\nu) = \sum c_j \mathcal{F}_{\mathfrak{g}}(\text{Ind}_P^G[\mu(\mathcal{O}_{j,\mathfrak{m}})])$ .

**4.14. Relation between AC and AS.** According to results of Schmid-Vilonen [SV2], the nilpotent  $G$ -orbits and  $K_{\mathbb{C}}$ -orbits in the formulas for AC and AS correspond via the Kostant-Sekiguchi correspondence, and  $c_i = \dim \chi_i$ .

**4.15.** The comparison between the Liouville measures of induced nilpotent orbits and the inducing data is done in [B3]. The analysis of the distributions  $\theta(f_P)$  is done in [B4] formula (8.3). Let  $AS(\pi_M) = \sum_j c_j \mathcal{O}_{j,\mathfrak{m}}$ . For each orbit  $\mathcal{O}_{j,\mathfrak{m}}$  write  $v_{ij} + X_{ij}$  for representatives of the orbits intersecting  $\mathcal{O}_{j,\mathfrak{m}} + \mathfrak{n}$  in open sets. Let  $C_G(v_{ij})$  and  $C_P(v_{ij})$  be the centralizers. Then

$$(34) \quad AS(\text{Ind}_P^G \pi_M) = \sum_{i,j} c_j \left| \frac{C_G(v_{ij})}{C_P(v_{ij})} \right| \mathcal{O}_{ij}.$$

## 5. COMPLEX GROUPS

**5.1.** We specialize the results in section 4.12 to 4.15 to the complex case. The main simplifications are that  $AC(\pi) = c_\pi \mathcal{O}_{\mathfrak{m}}$ , and there is only one  $\mathcal{O}$  which intersects  $\mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$  in a dense open set. We use AC for both the asymptotic cycle and support identified via the Kostant-Sekiguchi correspondence. Formula (34) becomes

$$(35) \quad AC(\text{Ind}_P^G[\pi]) = c_\pi \left| \frac{C_G(v)}{C_P(v)} \right| \mathcal{O}.$$

**5.2. Proof of theorem 4.10.1.** Since these results are clear for type A, we deal with types B, C, D only. We use the notation and parametrization in section 2.3. Character identities are consequences of theorem III in [BV2] and its applications as detailed in [B1].

Assume first that  $\mathcal{O}$  is triangular corresponding to  $\{e, h, f\}$ . Let  $\{e^\vee, h^\vee, f^\vee\}$  be the dual nilpotent orbit in  $\vee \mathfrak{g}$ . Let  $M(h)$  be the centralizer of  $h$ ,  $M(h^\vee) \subset G$  the centralizer of  $h^\vee$ . By section 9 in [BV2] on triangular nilpotent orbits, every unipotent representation is induced irreducible from a character of  $\chi$  of  $M(h)$ . Parametrize the representations by these characters,  $\chi \in \widehat{M(h)} \longleftrightarrow X_\chi$ . From [BV2], the passage from this parametrization to

the one given by characters of the component group of the dual nilpotent orbit is known explicitly. By [V1],

$$X_\chi|_{K=K} = R(\mathcal{O})_{\rho(\chi)} - Y_\chi$$

where  $Y_\chi$  is a genuine  $K$ -module.  $\rho(\chi)$  is a representation of the component group of the centralizer of  $e$ , trivial on the unipotent radical because it is algebraic. Since  $\dim \rho(\chi)$  is also the multiplicity, it follows that  $\rho(\chi)$  must be 1-dimensional. Since the reductive part of the centralizer of  $e$  is a product of classical groups,  $\rho(\chi)$  is trivial on the connected component. Thus  $\rho(\chi)$  is a character of  $A(\mathcal{O})$ .

On the other hand, again by [BV2],

$$\text{Ind}_{M(h^\vee)}^G[\text{Triv}] = \sum X_\chi.$$

Using Proposition 4.6.1, we get an identity

$$\sum R(\mathcal{O})_\psi = \sum X_\chi|_{K=K} = \sum R(\mathcal{O})_{\rho(\chi)} - \sum Y_\chi.$$

It follows that

$$X_\chi|_{K=K} = R(\mathcal{O})_{\rho(\chi)}.$$

It is clear that if  $X_\chi$  is the spherical unipotent representation, then  $\rho(\chi) = \text{Triv}$ .

Now let  $\mathcal{O}$  be a *special stably trivial* nilpotent orbit,  $\mathcal{O} \subset \mathfrak{g}(n)$ . The results in [BV2] imply that there is a 1-1 correspondence between characters of  $A(\mathcal{O})$  and unipotent representations. Choose an arbitrary parametrization of the unipotent representations by characters of  $A(\mathcal{O})$ , the trivial character should correspond to the spherical module. As before, for each unipotent representation  $X_\nu$ , there is a representation  $\rho(\nu)$  of the full centralizer of  $e \in \mathcal{O}$ , such that

$$(36) \quad X_\nu = R(\mathcal{O})_{\rho(\nu)} - Y_\nu,$$

with  $Y_\nu$  a genuine  $K$ -module.

Let  $\mathfrak{m} = \mathfrak{g}(n) \times \mathfrak{gl}(k_1) \times \cdots \times \mathfrak{gl}(k_r)$  be a Levi component of a parabolic subalgebra in  $\mathfrak{g}^+ := \mathfrak{g}(n + k_1 + \cdots + k_r)$ . There are  $k_1, \dots, k_r$  such that the orbit

$$(37) \quad \mathcal{O}^+ = \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}^+}[\mathcal{O} \times \text{triv} \times \cdots \times \text{triv}]$$

is *triangular*. Inducing  $X_\nu$  up to  $\mathfrak{g}^+$ , and using the decomposition formulas for such modules from [BV2] combined with Propositions 4.5.1 and 4.6.1, we conclude as before that  $\text{Ind } Y_\nu = 0$  so  $Y_\nu = 0$ , and the multiplicity of  $X_\nu$  is 1. Thus  $\rho(\nu)$  is a character of the component group  $A(\mathcal{O})$ , and counting occurrences in the induced modules, we conclude that the correspondence  $\nu \longleftrightarrow \rho(\nu)$  is 1-1. In other words, there is a parametrization  $\nu \longleftrightarrow X_\nu$  such that

$$X_\nu = R(\mathcal{O})_\nu.$$

**5.3. The correspondence  $\psi \longleftrightarrow X_\psi$ .** We give details for type C; the other types are similar. From section 2.3 we know that the unipotent representations are indexed by  $(\epsilon_0, \dots, \epsilon_k)$ , with  $\epsilon_j = \pm$  one for each pair of columns  $(c_{2j}, c_{2j+1})$ . The component group also has  $k + 1$  components,  $A(\mathcal{O}) \cong \mathbb{Z}_2^{k+1}$ , one for each even size of rows. The sizes of even rows are  $(r_0, \dots, r_k)$ . A character of  $A(\mathcal{O})$  is given by an  $(\eta_0, \dots, \eta_k)$ , with  $\eta_j = \pm$  according to whether the character is trivial or not on the corresponding  $\mathbb{Z}_2^j$ . It is enough to give the correspondence for the cases when all  $\eta_i = +$  except for one  $\eta_j = -$ . The matching is that one sets all the  $\epsilon_s = -$  for the pairs of columns with label larger than or equal to  $j$ . The following Corollary is key.

**Corollary 5.3.1.** *Let  $\mathfrak{m} \subset \mathfrak{g}$  be a Levi component,  $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m}$  a stably trivial special orbit, and  $\mathcal{O} = \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{m}}$  also stably trivial special. Then*

$$\text{Ind}_{\mathfrak{m}}^{\mathfrak{g}} X_{\mathfrak{m}, \nu} = \sum [\psi ; \nu] X_{\mathfrak{g}, \psi}$$

**Example 5.3.2.** *Consider the nilpotent orbit  $\mathcal{O} = (4422)$ . The unipotent representations are (writing  $\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix}$  for the parameter)*

$$(38) \quad \begin{aligned} \pi(4_+, 2_+) & \quad \begin{pmatrix} 2 & 1 & 0 & 1 & ; & 1 & 0 \\ 2 & 1 & 0 & 1 & ; & 1 & 0 \end{pmatrix} \\ \pi(4_-, 2_+) & \quad \begin{pmatrix} 2 & 1 & 0 & 1 & ; & 1 & 0 \\ 1 & 0 & -1 & -2 & ; & 1 & 0 \end{pmatrix} \\ \pi(4_+, 2_-) & \quad \begin{pmatrix} 2 & 1 & 0 & 1 & ; & 1 & 0 \\ 2 & 1 & 0 & 1 & ; & 0 & -1 \end{pmatrix} \\ \pi(4_-, 2_-) & \quad \begin{pmatrix} 2 & 1 & 0 & -1 & ; & 1 & 0 \\ 1 & 0 & -1 & -2 & ; & 0 & -1 \end{pmatrix} \end{aligned}$$

The labeling  $4_{\pm}, (2_{\pm})$  indicates the  $\epsilon$  on the columns of size 2 and 4 respectively.

Write the rings of regular functions as  $R(4^+2^+), R(4^-2^+), R(4^+2^-), R(4^-2^-)$ . Here the  $(4^{\pm}2_{\pm})$  indicate the  $\epsilon$  on the rows of size 4 and 2 respectively. Note that  $\mathcal{O} = (4422)$  is induced from  $(2211) \times \text{triv}$  of  $\text{sp}(4) \times \text{gl}(3)$  and also from  $(3322) \times \text{triv}$  of  $\text{sp}(10) \times \text{gl}(1)$ . The partitions denote rows.

The composition series are

$$\begin{aligned}
 & \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \otimes \text{triv} \longrightarrow \pi(4_+ 2_+) + \pi(4_- 2_-) \\
 & R(2^+ 1) \longrightarrow R(4^+ 2^+) + R(4^+ 2^-) \\
 & \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \otimes \text{triv} \longrightarrow \pi(4_+ 2_-) + \pi(4_- 2_+)
 \end{aligned}
 \tag{39}$$

$$R(2^- 1) \longrightarrow R(4^- 2^+) + R(4^- 2^-)$$

and

$$\begin{aligned}
 & \begin{pmatrix} 2 & 1 & 0 & -1 & ; & 1 \\ 2 & 1 & 0 & -1 & ; & 1 \end{pmatrix} \otimes \text{triv} \longrightarrow \pi(4_+ 2_+) + \pi(4_+ 2_-) \\
 & R(32^+) \longrightarrow R(4^+ 2^+) + R(4^- 2^+) \\
 & \begin{pmatrix} 2 & 1 & 0 & -1; & 1 \\ 1 & 0 & -1 & -2 & ; & 1 \end{pmatrix} \otimes \text{triv} \longrightarrow \pi(4_- 2_+) + \pi(4_- 2_-) \\
 & R(32^-) \longrightarrow R(4^+ 2^-) + R(4^- 2^-)
 \end{aligned}
 \tag{40}$$

In these formulas, the nilpotent (2211) was abbreviated as (21) with signs corresponding to the character on the rows of size 2, and (3322) was abbreviated as (32) with signs corresponding to the character on the rows of size 2.

## 6. THE KRAFT-PROCESI MODEL

**6.1. Basic Setup.** We follow [Bry]. Let  $\mathcal{O}$  be a nilpotent orbit given in terms of the columns of its partition. Remove a column. The resulting partition corresponds to a nilpotent orbit in a smaller classical Lie algebra. The type alternates  $C$  and  $B/D$ . We get a sequence  $(\mathfrak{g}_i, K_i)$  of classical algebras such that each  $((\mathfrak{g}_i, K_i), (\mathfrak{g}_{i+1}, K_{i+1}))$  is a dual pair. Furthermore each pair is equipped with an oscillator representation  $\Omega_i$  which gives the Theta correspondence. Form the  $(\mathcal{G}, K) := (\mathfrak{g}_0, K_0) \times \cdots \times (\mathfrak{g}_\ell, K_\ell)$ -module

$$\Omega := \bigotimes \Omega_i.$$

The representation we are interested in, is the  $(\mathfrak{g}_0, K_0)$ -module

$$\Pi = \Omega / (\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_\ell)(\Omega).$$

Let  $(\mathfrak{g}^1, K^1) := (\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_\ell, K_1 \times \cdots \times K_\ell)$ , and let  $\mathfrak{g}^1 = \mathfrak{k}^1 + \mathfrak{p}^1$  be the Cartan decomposition.

The following facts are standard.  $\Pi$  is an admissible  $(\mathfrak{g}_0, K_0)$ -module. It has an infinitesimal character compatible with the  $\Theta$ -correspondence, Proposition 3.3.1. Furthermore the  $K_i$  which are orthogonal groups are disconnected, so the nontrivial component group  $\mathcal{K}^1 := K^1/(K^1)^0$  still acts, and commutes with the action of  $(\mathfrak{g}_0, K_0)$ . Thus  $\Pi$  decomposes

$$\Pi = \bigoplus \Pi_\Psi$$

where  $\Pi_\Psi := \text{Hom}_{\mathcal{K}^1}[\Pi, \Psi]$ . The characters of  $\mathcal{K}^1$  are in 1-1 correspondence with the characters of  $A(\mathcal{O})$  as in section 5.3.

**6.2. The Main Result.** In the case of the representations at the beginning of Section 3.5, the ensuing representations are unipotent. Let  $\mathcal{V} := \coprod \text{Hom}[V_i, V_{i+1}]$ . This can be identified with a Lagrangian. Consider the variety  $\mathcal{Z} = \{(A_0, \dots, A_\ell)\} \subset \mathcal{V}$  given by the equations

$$A_i^* \circ A_i - A_{i+1} \circ A_i^* = 0, \dots, A_{\ell+1} \circ A_\ell^* = 0, \quad i = 0, \dots, \ell - 1.$$

**Theorem 6.2.1** ([Bry]).  *$\Omega$  has a  $(\mathcal{G}, K)$  compatible filtration so that*

$$\text{gr}(\Omega/\mathfrak{p}^1\Omega) \cong R(\mathcal{Z}).$$

*$K^1$  still acts, and in particular*

$$\text{gr}(\Omega/\mathfrak{g}^1\Omega)_{\mathcal{K}^1} \cong \text{gr}(\Omega/\mathfrak{p}^1\Omega)_{K^1} \cong R(\overline{\mathcal{O}}).$$

Consider the coinvariants  $R(\mathcal{Z})_{\mathfrak{k}^1}$ . Then  $\mathcal{K}^1 := K^1/(K^1)^0$  acts, and so we conclude

$$R(\mathcal{Z})_{\mathfrak{k}^1} = \bigoplus_{\Psi \in \widehat{\mathcal{K}^1}} R(\mathcal{Z})_\Psi$$

**Corollary 6.2.2.** *Assume the nilpotent orbit  $\mathcal{O}$  satisfies the conditions at the beginning of Section 3.5. Then*

$$\Pi(\mathcal{O}, \psi) |_{K^1} \cong R(\mathcal{Z})_\psi.$$

**Remark 6.2.3.**  *$A(\mathcal{O})$  does not act on  $\overline{\mathcal{O}}$ , so we cannot identify  $[R(\mathcal{Z})_{\mathfrak{k}^1} : \psi]$  with  $R(\overline{\mathcal{O}})$  as in the case  $\psi = \text{Id}$ . But  $K^1$  does act on  $\mathcal{Z}$ , so that the formula in the Corollary makes sense. In the cases when  $\overline{\mathcal{O}}$  is normal, it is reasonable to conjecture that  $R(\mathcal{O}, \psi) \cong [R(\mathcal{Z})_{\mathfrak{k}^1} : \psi]$ . This would follow from the conjecture that any regular function on the inverse image of  $\mathcal{O} \times \mathcal{O}_1 \times \dots \times \mathcal{O}_\ell$  is regular on all of  $\mathcal{Z}$ .*

## 7. BEYOND THE THETA CORRESPONDENCE

**7.1.** We consider the case of the *Spin* groups of type  $B_n$   $D_n$ . We are concerned with *genuine* unipotent representations. In coordinates this means that the  $K$ -types have half integer entries only.

**Theorem 7.1.1.** *A genuine representation  $(\pi, V)$  is unitary only if it is induced from a representation  $\pi_1 \otimes \dots \otimes \pi_k \otimes \pi_0$  on a Levi component  $L = GL(m_1) \times \dots \times GL(m_k) \times G_0$  where*

- (1) the representations  $\pi_i$  for  $i = 1, \dots, k$  are unitary with 1-dimensional lowest  $K$ -types  $(\mu_i + 1/2, \dots, \mu_i + 1/2)$  with  $\mu_i \in \mathbb{N}$ ,
- (2)  $\pi_0$  has lowest  $K$ -type spin.

*Proof.* This is a standard *bottom layer* argument. See [Br] for this specific case, and [B1] for the more general complex case.  $\square$

It is conjectured that the basic cases from which the unitary dual is obtained via unitary induction and complementary series are such that  $\pi_0$  is unitary, and the infinitesimal character is integral for a system of type  $C_n \times C_n$  for type  $B$  (coroots in the Langlands dual), and  $D_n \times D_n$  for type  $D$ . We therefore concentrate on representations with lowest  $K$ -type spin. The following is a sharper conjecture about the basic cases, following the parametrization in 3.5. We treat type  $B$  in detail, case  $D$  is analogous.

**7.2. Type B.** The orbit  $\mathcal{O}$  has columns  $(m'_0, \dots, m'_{2p})$  and let

$$(41) \quad (m_0)(m_1, m_2) \dots (m_{2p-1}, m_{2p}) \quad m_{2i} = m_{2i+1} + 1.$$

The columns satisfying  $m'_{2j} = m'_{2j+1}$  were removed. The parameter

$$(42) \quad \begin{aligned} m_{2j} \text{ odd} &\longleftrightarrow \begin{pmatrix} \frac{m_{2j}-1}{2} & \dots & 1 & \frac{-m_{2j}+2}{2} & \dots & \frac{-1}{2} \\ \frac{m_{2j}-2}{2} & \dots & \frac{1}{2} & \frac{-m_{2j}+1}{2} & \dots & -1 \end{pmatrix} \\ m_{2j} \text{ even} &\longleftrightarrow \begin{pmatrix} \frac{m_{2j}}{2} & \dots & \frac{1}{2} & \frac{-m_{2j}+1}{2} & \dots & \frac{-1}{2} \\ \frac{m_{2j}-1}{2} & \dots & 0 & \frac{-m_{2j}+1}{2} & \dots & 0 \end{pmatrix} \\ m'_{2j} = m'_{2j+1} &\longleftrightarrow \begin{pmatrix} \frac{m'_{2j}}{2} & \dots & \frac{-m'_{2j}}{2} & \frac{-m_{2j}+1}{2} & \dots & \frac{m'_{2j}+1}{2} \\ \frac{m'_{2j}}{2} & \dots & \frac{-m'_{2j}-1}{2} & -m'_{2j} & \dots & \frac{m'_{2j}}{2} \end{pmatrix} \end{aligned}$$

is genuine. The infinitesimal character is  $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ , same as in 3.5, but arranged so that  $\begin{pmatrix} \lambda \\ w\lambda \end{pmatrix}$  has lowest  $K$ -type spin. As before, the  $m'_{2j} = m'_{2j+1}$  give rise to complementary series, and we concentrate on the case when there are no such pairs. Note that the orbit  $\mathcal{O}$  has an arbitrary number of rows of even size, while the odd sized rows are  $1, 3, 5, \dots, 4k+1$ .

The integral system for this parameter (the coroots with integral inner product with the parameter) form a system of type  $C \times C$ . The corresponding *endoscopic group* is type  $B \times B$ .

**Proposition 7.2.1.** *There is a unique genuine parameter with infinitesimal character  $\lambda_{\mathcal{O}}$  given by (42) and associated cycle a multiple of  $\mathcal{O}$  as in (41). (42).*

*Proof.* We use the generalized Kazhdan-Lusztig conjectures. It is enough to consider one of the factors,  $C_n$  in the integral roots of type  $C_n \times C_n$ . The left and right maximal primitive ideals for part of  $\lambda_L$  and  $w\lambda_R$  correspond to what are called the Springer and the Lusztig primitive ideal cell for the same nilpotent orbit. These do not have any Weyl group representations in

common except for the special one, occurring with multiplicity 1. This is the uniqueness of the parameter. The rest of the argument is as in [B1].  $\square$

Denote by  $A_{Spin}(\mathcal{O})$  the component group of the centralizer of an  $e \in \mathcal{O}$  in the  $Spin$ -group. Recall that  $A(\mathcal{O}) = \mathbb{Z}_2^{2k}$ .

**Proposition 7.2.2.**  *$A_{Spin}(\mathcal{O})$  is a nontrivial extension of  $A(\mathcal{O})$  by  $\mathbb{Z}_2$ :*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow A_{Spin}(\mathcal{O}) \longrightarrow A(\mathcal{O}) \longrightarrow 1.$$

*In particular,  $A_{Spin}(\mathcal{O})$  has  $2^{2k}$  characters lifted from  $A(\mathcal{O})$ , and one genuine character of degree  $2^k$ .*

*Proof.* Let  $(V, Q)$  be a quadratic space of dimension  $2r + 1$  with a basis  $\{e_i, v, f_i\}$  satisfying  $Q(e_i, f_j) = \delta_{ij}$ ,  $Q(e_i, v) = Q(f_j, v) = Q(e_i, e_j) = Q(f_i, f_j) = 0$ , and  $Q(v, v) = -1$ . Let  $C(V)$  be the Clifford algebra with automorphisms  $\alpha$  defined by  $\alpha(x_1 \dots x_r) = (-1)^r x_1 \dots x_r$  and  $\star$  given by  $(x_1 \dots x_r)^\star = (-1)^r x_r \dots x_1$ . The double cover of  $O(V)$  is

$$Pin(V) := \{x \in C(V) \mid x \cdot x^\star = 1, \alpha(x)Vx^\star \subset V\},$$

and the double cover of  $SO(V)$  by the elements in  $Pin(V)$  which are in  $C(V)^{even}$ . The action of  $Pin(V)$  on  $V$  is given by  $\rho(x)v = \alpha(x)vx^\star$ . The element  $-I \in O(V)$  is covered by

$$(43) \quad \pm E_{2r+1} = \pm v \prod_{1 \leq i \leq r} [(1 - e_i f_i) / \sqrt{-1}].$$

Suppose  $V = V_{2i+1} \oplus V_{2j+1}$  is a quadratic space and direct sum of spaces of dimensions  $2i + 1, 2j + 1$  so that the restriction of the quadratic form is nondegenerate on each of them. Then there are two such operators,  $E_{2i+1}$  and  $E_{2j+1}$ . They satisfy the relations

$$\begin{aligned} E_{2i+1} E_{2j+1} &= -E_{2j+1} E_{2i+1} \\ E_{2r+1}^2 &= (\sqrt{-1})^{-r}. \end{aligned}$$

Fix an element  $\varepsilon \in \mathcal{O}$ . Its action on  $V$  can be described in terms of Jordan blocks. Because  $\varepsilon$  is skew with respect to  $Q$ , the action on an odd sized block can be represented by a sequence of arrows

$$e_1 \longrightarrow e_2 \longrightarrow \dots \longrightarrow e_r \longrightarrow v \longrightarrow f_r \longrightarrow -f_{r-1} \longrightarrow \dots \longrightarrow (-1)^{r+1} f_1 \longrightarrow 0,$$

where the  $e_i, f_j$  are in duality and  $v$  has norm 1. The group  $A(\mathcal{O})$  is generated by even products of elements each of which act by  $-I$  on one of the odd *Jordan blocks* of  $\mathcal{O}$ , and  $+I$  on the others. The inverse image of  $A(\mathcal{O})$  in  $Spin(V)$  is generated by even products of  $\pm E_{2r+1}$  as in (43).  $\square$

**Proposition 7.2.3.** *The parameters in (42) are unitary.*

*Proof.* See [Br].  $\square$

Consider the special case of  $\mathcal{O}$  with columns  $(2m+1, 2m)$ . The parameter is

$$(44) \quad \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} m, & \dots & 1 & -1/2, & \dots & -m+1/2 \\ m-1/2 & \dots & 1/2 & -1 & \dots & -m \end{pmatrix}$$

The orbit  $\mathcal{O}$  has  $SL_2$ -triple  $\{E, h, F\}$  with  $h = (1, \dots, 1)$  and  $E$  with Jordan blocks

$$e_i \longrightarrow -f_i \longrightarrow 0.$$

Let  $\mathfrak{p} := \mathfrak{m} + \mathfrak{n} = C_h(0) + C_h(1) + C_h(2)$  be the parabolic subalgebra corresponding to  $h$ , where  $C_h(i)$  are the  $i$ -eigenspaces of  $h$ . In particular  $C_h(0) = \mathfrak{m} \cong gl(2m)$ , and  $\mathfrak{n} = C_h(1) + C_h(2)$ . The centralizer of  $E$  is  $C_E = C_E(0) + C_h(1) + C_h(2)$ , with  $C_E(0) \cong sp(2m, \mathbb{C}) \subset gl(2m)$  embedded in the standard way. The component group of the centralizer of  $E$  in  $SO(4m+1)$  is trivial, while the centralizer in  $Spin(4m+1, \mathbb{C})$  is  $\mathbb{Z}_2$ . So there are two characters of  $A_{Spin}(\mathcal{O})$ ,  $\psi_{triv}$  and  $\psi_{gen}$ .

**Proposition 7.2.4.** *Let  $V(\mu)$  denote a  $K$ -type with highest weight  $\mu$ .*

$$R(\mathcal{O}, \psi_{triv}) = \sum V(a_1, a_1, \dots, a_m, a_m),$$

$$R(\mathcal{O}, \psi_{gen}) = \sum V(a_1 + 1/2, a_1 + 1/2, \dots, a_m + 1/2, a_m + 1/2).$$

with  $a_1 \geq \dots \geq a_m \geq 0$ .

*Proof.* Kostant's theorem implies that the  $\mathfrak{n}$  fixed vectors of  $V(\mu_1, \dots, \mu_{2m})$  are the  $gl(2m, \mathbb{C})$ -module generated by the highest weight. The vectors fixed by  $sp(2m, \mathbb{C})$  follow by Helgason's theorem.  $\square$

**Corollary 7.2.5.**

$$X(\mathcal{O}, triv) |_K \cong R(\mathcal{O}, \psi_{triv}), \quad X(\mathcal{O}, gen) |_K \cong R(\mathcal{O}, \psi_{gen})$$

*Proof.* The first identity follows from the Theta Correspondence,  $X(\mathcal{O}, triv)$  matches the trivial representation on  $Sp(2m, \mathbb{C})$ . It also follows from the arguments in [McG1]. An extension of this argument implies the second identity, noting that  $Spin \otimes Spin$  is a fine  $K$ -type for the appropriate cover of  $So(2m+1, 2m)$ . In more detail, the character formula for  $X(\mathcal{O}, gen)$  is

$$X(\mathcal{O}, gen) = \sum_{w \in W(B_n \times B_n)} \epsilon(w) X(w \cdot (\lambda_L, \lambda_R), (\lambda_L, \lambda_R)).$$

Using induction in stages and restricting to  $K$ , this matches the formula for induction from  $Spin \otimes Spin$  to  $S[Pin(2m+1) \times Pin(2m)]$  to  $Spin(4m+1)$ . Then pass to the real form, and note that  $Spin \otimes Spin$  is a fine  $K$ -type.  $\square$

**Proposition 7.2.6.** *The multiplicity of  $\mathcal{O}$  in  $X(\mathcal{O}, \psi_{gen})$  in Equation (42) is  $2^p$ . Let  $\psi_{gen}$  be the unique irreducible representation of dimension  $2^p$  of  $A_{Spin}(\mathcal{O})$ . Then*

$$X(\mathcal{O}, \psi_{gen}) |_K = R(\mathcal{O}, \psi_{gen})$$



*Proof.* The proof is essentially the same as for the cases in Section 5.2. The triangular orbits are replaced by the orbits with rows  $(1, 3, \dots, 4k + 1)$ . The induced modules from the two parabolic subalgebras are both irreducible. The induced from the parabolic subalgebra with Levi component products of  $GL$  gives multiplicity  $2^p$ . For the induced from the other parabolic subalgebra, the trivial representation is replaced by the representation with orbit  $\mathcal{O}$  corresponding to the columns  $(2m + 1, 2m)$ . Since  $X(\mathcal{O}, gen)$  is genuine, and  $2^p$  is the smallest possible for a genuine representation of  $C_{Spin}(\mathcal{O})$  (this representation is trivial on the connected component), the proof from Section 5.2 carries over. We omit further details.  $\square$

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